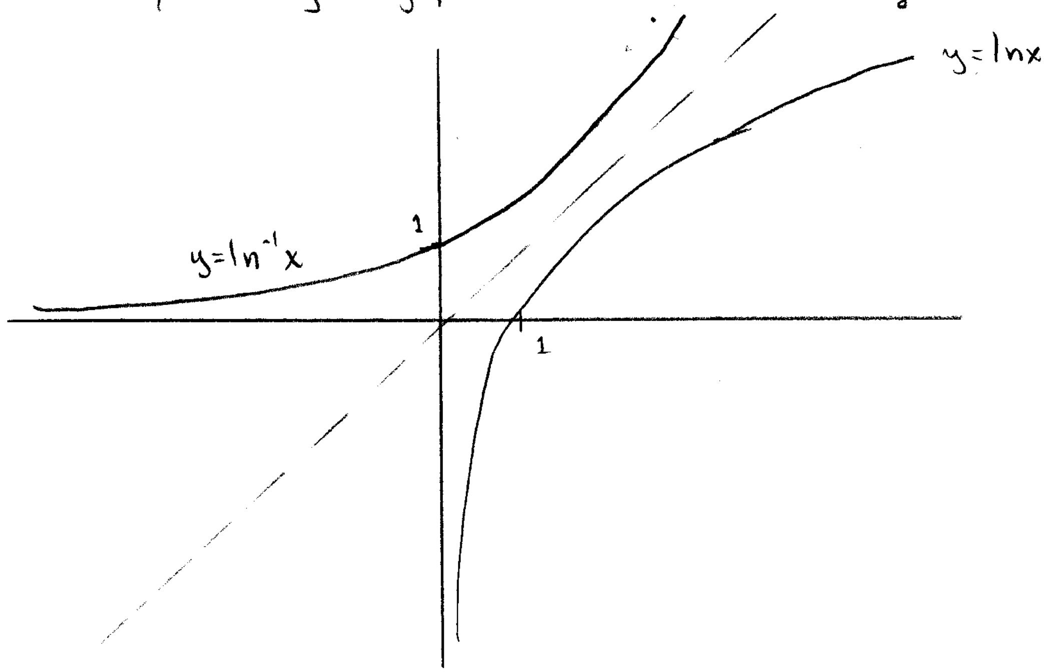
APPM 1350: Section 6.3: The Exponential Function

In section 6.1 we discussed inverse functions and noted that a necessary and sufficient condition for $f(x)$ to have an inverse $f^{-1}(x)$ is that $y=f(x)$ be one-to-one. This means that for each x there is a unique y value and visa versa. Then in section 6.2 we discussed the natural logarithm, defined as:

$$\ln x = \int_1^x \frac{dt}{t}$$

We noted that the logarithm is only defined for $x>0$. It is a one-to-one mapping, hence the logarithm has an inverse $\ln^{-1} x$. We can plot this inverse by reflecting the graph of $\ln x$ across the line $x=y$:



Note that the domain of $\ln^{-1} x$ is the range of $\ln x$, i.e., $(-\infty, \infty)$ whereas the range of $\ln^{-1} x$ is the domain of $\ln x$, i.e., $(0, \infty)$. We assign a particular value to $\ln^{-1} 1$, namely;

$$\boxed{e = \ln^{-1} 1}$$

whose value is approximately $2.71828\ldots$

Now consider $e = \ln^{-1} 1$ raised to a power. So

$$e^x = (\ln^{-1} 1)^x$$

Taking the logarithm of both sides and manipulating, we find:

$$\ln e^x = \ln\left[(\ln^{-1} 1)^x\right]$$

$$= x \ln(\ln^{-1} 1)$$

$$= x \qquad \text{since } \ln \text{ and } \ln^{-1} \text{ are inverses.}$$

So we have

$$\ln e^x = x$$

$$\Rightarrow \ln^{-1}(\ln e^x) = \ln^{-1} x$$

$$\Rightarrow e^x = \ln^{-1} x$$

So the number $e = \ln 1$ raised to the x power is in fact the inverse of the natural logarithm.

Def For $\forall\, x \in \mathbb{R}$, $e^x = \ln^{-1} x$

Important Rules About $\ln x$ and e^x

Inverse equations:	$e^{\ln x} = x$	for all $x > 0$
	$\ln(e^x) = x$	for all x

Examples

$$\ln e^{\pi} = \pi \ln e = \pi$$

$$e^{\ln \pi} = \pi$$

$$\ln e^{-1} = -\ln e = -1$$

$$\ln e^{x^3} = x^3 \ln e = x^3$$

More generally we are interested in equations involving natural logs and exponentials. For example:

Example

(#6, sec 6.3) Solve $\ln y = -t+5$ for y in terms of t.

Solution:

$$\ln y = -t+5$$

$$\Rightarrow e^{\ln y} = e^{-t+5}$$

$$\Rightarrow y = e^{-t+5}$$

Example

(#12c, sec 6.3) Solve for k in $e^{(\ln 0.8)k} = 0.8$

Solution:

$$e^{(\ln 0.8)k} = 0.8$$

$$\Rightarrow \ln e^{(\ln 0.8)k} = \ln 0.8$$

$$\Rightarrow (\ln 0.8)k = \ln 0.8$$

$$\Rightarrow k = 1$$

These two examples illustrate the following useful rules:

- To remove logarithms from an equation, exponentiate both sides
- To remove exponentials, take the logarithm of both sides

The first rule applies to equations of the form $\ln y = f(x)$ while the second rule applies to equations of the form $e^y = f(x)$, as well as similar variants.

Law of Exponents

For all numbers x, x_1, and x_2

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

These rules follow from the properties of logarithms (see the proof of #1 on page 469 of Thomas).

Example
(#16, sec 6.3) Solve for t given $e^{(x^2)}e^{(2x+1)} = e^t$

Solution:

$$e^{(x^2)}e^{(2x+1)} = e^t$$

$$\Rightarrow e^{x^2+2x+1} = e^t$$ (rule # 1 on previous page)

$$\Rightarrow \ln e^{x^2+2x+1} = \ln e^t$$

$$\Rightarrow t = x^2+2x+1$$

Example Find $(e^2)^{\ln x}$

Solution:

$$(e^2)^{\ln x} = e^{2\ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

Example Evaluate $e^{1-\ln x}$

Solution:

$$e^{1-\ln x} = e^1 e^{-\ln x}$$

$$= e \cdot e^{\ln x^{-1}}$$ since $e^1 = e$

$$= e\,\frac{1}{x}$$

$$= \frac{e}{x}$$

The Derivative and Integral of e^x

Consider:

$$y = e^x$$

$$\Rightarrow \ln y = x$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = y = e^x$$

In other words;

$$\boxed{\frac{d}{dx} e^x = e^x}$$

The derivative of e^x is itself! For function $u = u(x)$ we use the chain rule to get:

> If u is any differentiable function of x, then
>
> $$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

Further, since e^x is an antiderivative of itself,

$$\boxed{\int e^x \, dx = e^x + C}$$

<u>Example</u>
(#18, sec 6.3) Find dy/dx given $y = e^{2x/3}$

Solution:

$$\frac{d}{dx} y = \frac{d}{dx} e^{2x/3}$$

$$= e^{2x/3} \frac{d}{dx}\left(\frac{2x}{3}\right)$$

$$= \frac{2}{3} e^{2x/3}$$

<u>Example</u>
(#42, sec 6.3) Find $\int (2e^x - 3e^{-2x})\,dx$.

Solution:

$$\int (2e^x - 3e^{-2x})\,dx = 2\int e^x\,dx - 3\int e^{-2x}\,dx$$

Let $u = -2x$

$du = -2\,dx$

$$= 2\int e^x\,dx + \frac{3}{2}\int e^u\,du$$

$$= 2e^x + \frac{3}{2}e^u + c$$

$$= 2e^x + \frac{3}{2}e^{-2x} + c$$

Example

(#35, sec 6.3) Find dy/dx given $y = \int_0^{\ln x} \sin e^t \, dt$

Solution:

Using the fundamental theorem of calculus and the chain rule;

$$\frac{d}{dx} y = \frac{d}{dx} \int_0^{\ln x} \sin e^t \, dt$$

$$\Rightarrow \frac{dy}{dx} = \sin e^{\ln x} \frac{d}{dx} \ln x$$

$$= \frac{1}{x} \sin x$$

Remember, if $u = u(x)$:

$$\frac{d}{dx} \int_a^{u(x)} f(t)\, dt = f[u(x)] \frac{du}{dx}$$

Example

(#58, sec 6.3) Find $\int e^{\csc(\pi + t)} \csc(\pi + t) \cot(\pi + t) \, dt$

Solution:

$$\int e^{\csc(\pi+t)} \csc(\pi+t) \cot(\pi+t) dt$$

$$= -\int e^u \, du$$

$$= -e^u + c$$

$$= -e^{\csc(\pi+t)} + c$$

Let $u = \csc(\pi + t)$

$\Rightarrow \frac{du}{dt} = -\csc(\pi+t)\cot(\pi+t)$

Some Interesting Differential Equations

Example

(# 64, sec 6.3) Solve the IVP $\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t})$; $y(\ln 4) = \frac{2}{\pi}$

Solution:

$$\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t})$$

$$\Rightarrow dy = e^{-t}\sec^2(\pi e^{-t})\,dt$$

$$\Rightarrow \int dy = \int e^{-t}\sec^2(\pi e^{-t})\,dt$$

$$\Rightarrow y(t) = \int e^{-t}\sec^2(\pi e^{-t})\,dt$$

Let $u = \pi e^{-t}$

$du = -\pi e^{-t}\,dt$

$$= -\frac{1}{\pi}\int \sec^2 u\,du$$

$$= -\frac{1}{\pi}\tan u + c$$

$$= -\frac{1}{\pi}\tan(\pi e^{-t}) + c$$

Since $y(\ln 4) = \frac{2}{\pi}$, we have

$$\frac{2}{\pi} = -\frac{1}{\pi}\tan(\pi e^{-\ln 4}) + c$$

$$\Rightarrow \frac{2}{\pi} = -\frac{1}{\pi}\tan\left(\frac{\pi}{4}\right) + c$$

$$\Rightarrow \frac{2}{\pi} = -\frac{1}{\pi} + c$$

$$\Rightarrow c = \frac{3}{\pi}$$

Hence

$$y(t) = -\frac{1}{\pi}\tan(\pi e^{-t}) + \frac{3}{\pi}$$