Section 6.4: a^x and $\log_a x$

In the previous two sections we defined $\ln x$ and e^x and determined their derivatives and the indefinite integrals of $\frac{1}{x}$ and e^x. We now expand these concepts to do the same for a^x and $\log_a x$ for arbitrary $a > 0$.

The Function a^x

For any $a > 0$, consider

$$y = a^x$$

Then

$$\ln y = x \ln a$$

$$\Rightarrow y = e^{x \ln a}$$

Hence:

> **Def**: For any numbers $a > 0$ and any value of x;
>
> $$a^x = e^{x \ln a}$$

This gives us a way of calculating the derivative of a^x. Specifically

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$

$$= e^{x \ln a} \frac{d}{dx} x \ln a$$

$$= \ln a \, e^{x \ln a}$$

$$= a^x \ln a$$

More generally, using the chain rule we have

> If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
>
> $$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

Example
(#21, sec 6.4) Find $\frac{d}{dt} 2^{\sin 3t}$

Solution:

$$\frac{d}{dt} 2^{\sin 3t} = 2^{\sin 3t} (\ln 2) \frac{d}{dt} \sin 3t$$

$$= (\ln 2)\left(2^{\sin 3t}\right)(3 \cos 3t)$$

$$= 3(\ln 2)(\cos 3t)\left(2^{\sin 3t}\right)$$

The Integral of a^u

Assume $a \neq 1$ and $a > 0$. Then

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

$$\Rightarrow \frac{1}{\ln a} \frac{d}{dx} a^u = a^u \frac{du}{dx}$$

where it is important to remember to u is a differentiable function of x.

Then; swapping the left- and right-hand sides in the previous equation, we have:

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx} a^u$$

$$\Rightarrow \int a^u \frac{du}{dx}\, dx = \int \frac{1}{\ln a} \frac{d}{dx} a^u \, dx$$

$$\Rightarrow \int a^u du = \frac{1}{\ln a} \int \left(\frac{d}{dx} a^u\right) dx$$

$$= \frac{1}{\ln a}\left(a^u + c\right) .$$

So; for $a > 0$ and $a \neq 1$ and u a differentiable function of x;

$$\boxed{\int a^u du = \frac{a^u}{\ln a} + c}$$

Note something important. When we write du we are talking about the differential of a function $u(x)$. As usual;

$$du(x) = \left[\frac{d}{dx} u(x)\right] dx$$

$$= u'(x)\, dx$$

So what we really mean by this integral is:

$$\boxed{\int a^u du = \int a^u u' \, dx = \frac{a^u}{\ln a} + c}$$

This is of course what we did to derive the integral, just be sure you understand what du means in terms of dx.

Example
(#47, sec 6.4)

$$\int 5^x\,dx = \frac{5^x}{\ln 5} + c$$

In this case $u(x) = x$
$\Rightarrow du = dx$

Example
(#54, sec 6.4)

$$\int_0^{\pi/4} \left(\tfrac{1}{3}\right)^{\tan t} \sec^2 t\,dt = \int_0^1 \left(\tfrac{1}{3}\right)^u du$$

$$= \left.\frac{\left(\frac{1}{3}\right)^u}{\ln 3}\right|_0^1$$

$$= \frac{1}{\ln 3}\left[\frac{1}{3} - 1\right]$$

$$= \frac{2}{3\ln 3}$$

$u = \tan t$
$\Rightarrow du = \sec^2 t\,dt$
$t = 0 \Rightarrow u = \tan 0 = 0$
$t = \frac{\pi}{4} \Rightarrow u = \tan\frac{\pi}{4} = 1$

Example
(#55, sec 6.4)

$$\int_2^4 x^{2x}(1+\ln x)\,dx = \frac{1}{2}\int_{2^4}^{2^8} du$$

$$= \left.\frac{1}{2}u\right|_{2^4}^{2^8}$$

$$= \frac{1}{2}\left(2^8 - 2^4\right)$$

$$= 32{,}760$$

$u = x^{2x}$
$\Rightarrow \ln u = 2x\ln x$
$\Rightarrow \frac{1}{u}\frac{du}{dx} = 2\ln x + 2$
$\Rightarrow \frac{du}{dx} = 2x^{2x}(1+\ln x)$
$\Rightarrow du = 2x^{2x}(1+\ln x)\,dx$
Also: $x = 2 \Rightarrow u = 2^4$
$x = 4 \Rightarrow u = 4^8$

Logarithms with Base a

When we introduced the natural logarithm we defined it in terms of a definite integral and then introduced e^x as the inverse of $\ln x$. Here we go the opposite direction, we have a^x and define the logarithm to the base a of x as its inverse. Thus:

Def For any positive number $a \neq 1$

$$\log_a x = \text{inverse of } a^x$$

So by definition:

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$

where $a > 0$ and $a \neq 1$.

How do we actually evaluate $\log_a x$? Consider:

$$y = \log_a x$$

$$\Rightarrow a^y = x$$

$$\Rightarrow \ln a^y = \ln x$$

$$\Rightarrow y \ln a = \ln x$$

$$\Rightarrow y = \frac{\ln x}{\ln a}$$

$$\boxed{\Rightarrow \log_a x = \frac{\ln x}{\ln a}}$$

Note also that $\ln x = \log_e x$.

Example
(# 2, see 6.4)

a) $2^{\log_2 3} = 3$

b) $10^{\log_{10}(\frac{1}{2})} = \frac{1}{2}$

c) $\pi^{\log_\pi 7} = 7$

d) $\log_{11} 121 = \log_{11} 11^2 = \frac{1}{2}\log_{11} 11 = \frac{1}{2}$

e) $\log_{121} 11 = \log_{121} 121^{1/2} = \frac{1}{2}\log_{121} 121 = \frac{1}{2}$

f) $\log_3\left(\frac{1}{9}\right) = \log_3 3^{-2} = -2\log_3 3 = -2$

Note: $\log_a x$ obeys the same algebraic rules as $\ln x$. By the same token, a^x obeys the same algebraic rules as e^x

Example
(#8, see 6.4) Solve $8^{\log_8 3} - e^{\ln 5} = x^2 - 7^{\log_7(3x)}$ for x

Solution:

$$8^{\log_8 3} = 3$$

$$e^{\ln 5} = 5$$

$$7^{\log_7 3x} = 3x$$

So we have

$$3 - 5 = x^2 - 3x$$

$$\Rightarrow x^2 - 3x + 2 = 0$$

$$\Rightarrow (x-2)(x-1) = 0$$

$$\Rightarrow x = 1 \text{ or } x = 2$$

The Derivative of $\log_a u$

To determine $\frac{d}{dx}\log_a u$, convert it to a derivative of a natural log:

$$\frac{d}{dx}\log_a u = \frac{d}{dx}\frac{\ln u}{\ln a}$$

$$= \frac{1}{\ln a}\frac{d}{dx}\ln u$$

$$= \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}$$

Hence:

$$\boxed{\frac{d}{dx}\log_a u = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}}$$

Example

(#24, sec 6.4)

$$\frac{d}{d\theta}\log_3(1+\theta\ln 3) = \frac{1}{\ln 3}\frac{1}{1+\theta\ln 3}\frac{d}{d\theta}(1+\theta\ln 3)$$

$$= \left(\frac{1}{\ln 3}\right)\left(\frac{1}{1+\theta\ln 3}\right)\left(\ln 3\right)$$

$$= \frac{1}{1+\theta\ln 3}$$

Example

(#34, see 6.4)

$$\frac{d}{dx}\log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right) = \frac{d}{dx}\,\frac{\ln\left(\frac{x^2e^2}{2\sqrt{x+1}}\right)}{\ln 2}$$

$$= \frac{1}{\ln 2}\frac{d}{dx}\left\{\ln x^2 + \ln e^2 - \ln 2 - \ln\sqrt{x+1}\right\}$$

$$= \frac{1}{\ln 2}\frac{d}{dx}\left\{2\ln x + 2 - \ln 2 - \frac{1}{2}\ln(x+1)\right\}$$

$$= \frac{1}{\ln 2}\left\{\frac{2}{x} - \frac{1}{2(x+1)}\right\}$$

$$= \frac{1}{\ln 2}\left[\frac{4(x+1) - x}{2x(x+1)}\right]$$

$$= \frac{3x+4}{(2\ln 2)\,x(x+1)}$$

Integrals Involving $\log_a x$

As with derivatives, to compute integrals involving $\log_a x$, convert to natural logarithms.

Example

(#62, see 6.4)

$$\int_1^4 \frac{\log_2 x}{x}\,dx = \int_1^4 \frac{1}{x}\left(\frac{\ln x}{\ln 2}\right)dx$$

$$= \frac{1}{\ln 2}\int_1^4 \frac{\ln x}{x}\,dx$$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$x = 1 \Rightarrow u = \ln 1 = 0$

$x = 4 \Rightarrow u = \ln 4$

$$= \frac{1}{\ln 2}\int_0^{\ln 4} u\,du$$

$$= \frac{1}{\ln 2}\,\frac{u^2}{2} = \frac{(\ln 4)^2}{2\ln 2} = \frac{(\ln 4)^2}{(\ln 2^2)} = \frac{(\ln 4)^2}{(\ln 4)} = \ln 4$$

Example

(# 70, see 6.4)

$$\int \frac{dx}{x(\log_8 x)^2} = \int \frac{1}{x}\left(\frac{\ln 8}{\ln x}\right)^2 dx$$

$$= (\ln 8)^2 \int \frac{dx}{x(\ln u)^2}$$

Let $u = \ln x$

$\Rightarrow du = \frac{dx}{x}$

$$= (\ln 8)^2 \int \frac{du}{u^2}$$

$$= (\ln 8)^2 \left(-\frac{1}{u} + c\right)$$

$$= -\frac{(\ln 8)^2}{\ln x} + c$$

Common Logarithms

Base 10 logarithms are called common logarithms. Examples include earthquake intensity (Richter scale).

$$\text{Earthquake magnitude} = \log_{10}\left(\frac{a}{T}\right) + B$$

where a is the amplitude of ground motion in microns (10^{-6} m), at the point where the seismogram is recorded, T is the period of the seismic wave in seconds, and B is an empirically derived constant.

Other examples where common logs appear are in measurements of acidity (pH) and sound levels (decibals). See pages 479-480 for more discussion of this.