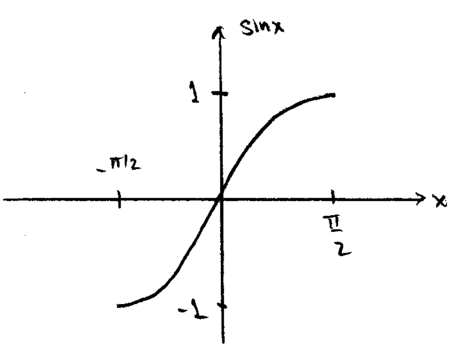
# APPM 1350: Section 6.8: Inverse Trigonometric Functions

Trig functions are not one-to-one. However, if we restrict their domains appropriately they can be made so. We there define the inverse trig functions by restricting the domains of the trig functions such that

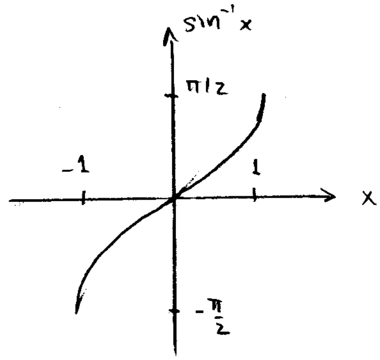
**Domain Restrictions That Make the Trigonometric Functions One-to-One**

| Function | Domain | Range |
|---|---|---|
| $\sin x$ | $[-\pi/2\ ,\ \pi/2]$ | $[-1, 1]$ |
| $\cos x$ | $[0, \pi]$ | $[-1, 1]$ |
| $\tan x$ | $(-\pi/2, \pi/2)$ | $(-\infty, \infty)$ |
| $\cot x$ | $(0, \pi)$ | $(-\infty, \infty)$ |
| $\sec x$ | $[0, \pi/2) \cup (\pi/2, \pi]$ | $(-\infty, \ -1] \cup [1, \infty)$ |
| $\csc x$ | $[-\pi/2, 0) \cup (0, \pi/2]$ | $(-\infty, \ -1] \cup [1, \infty)$ |

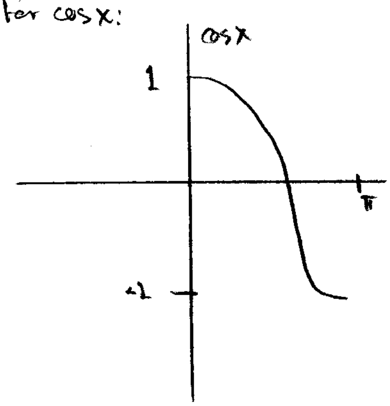
So for $\sin x$:



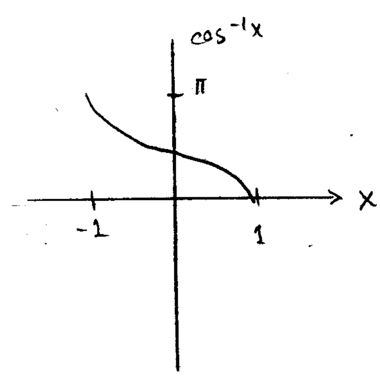

$\Rightarrow$

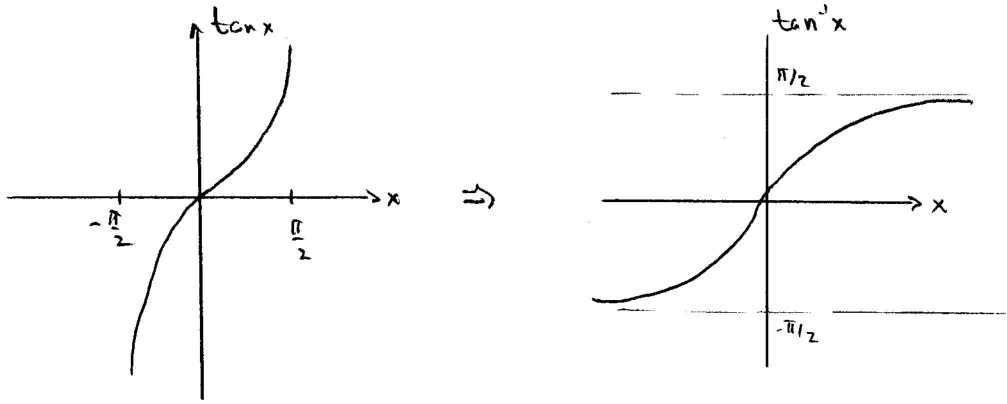



For $\cos x$:




$\Rightarrow$

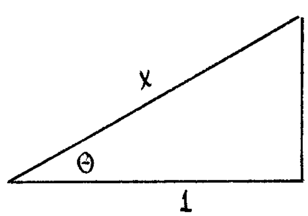





Note the following relationships

$$\boxed{\begin{aligned} \sec^{-1}x &= \cos^{-1}\left(\tfrac{1}{x}\right) \\ \csc^{-1}x &= \sin^{-1}\left(\tfrac{1}{x}\right) \\ \cot^{-1}x &= \tan^{-1}\left(\tfrac{1}{x}\right) \end{aligned}}$$

These follow trivially from drawing a triangle. For example, suppose we have $\theta = \sec^{-1}x$. Then since $\sec\theta$ is hypotenuse over adjacent:

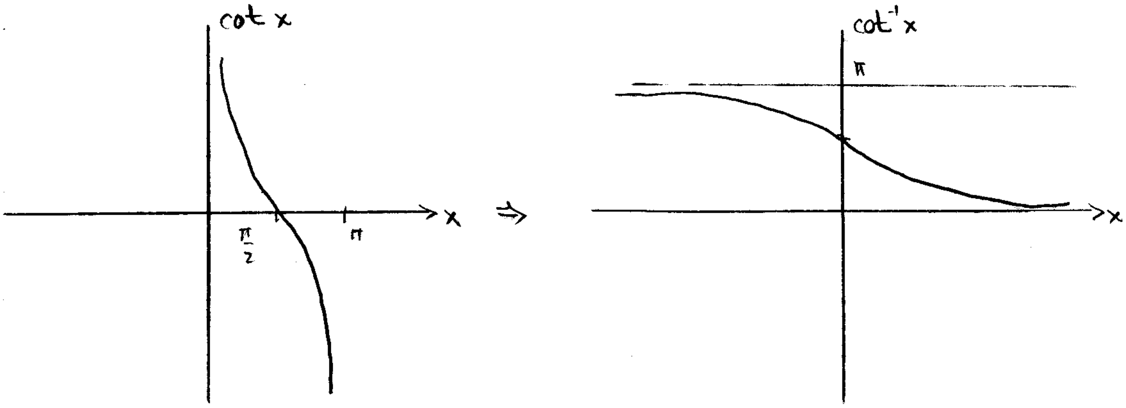


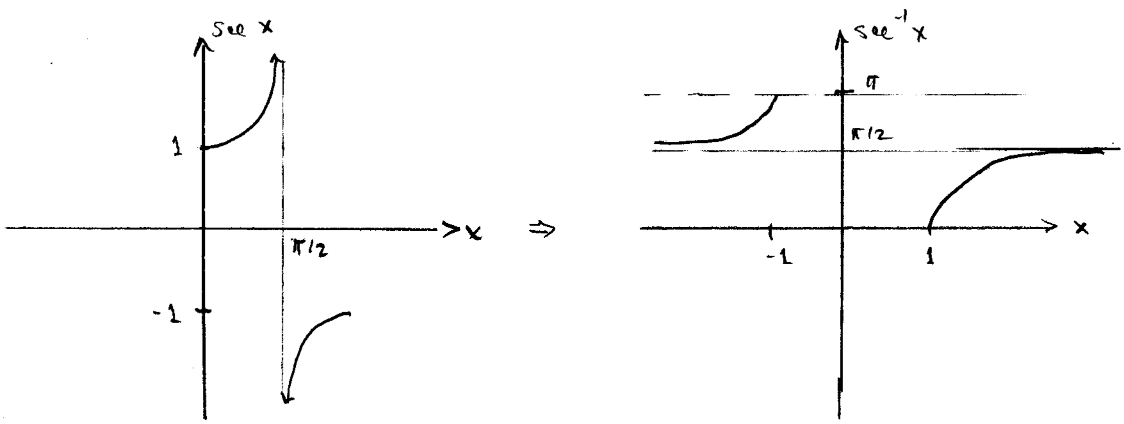

$$\cos\theta = \frac{1}{x} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{x}\right)$$

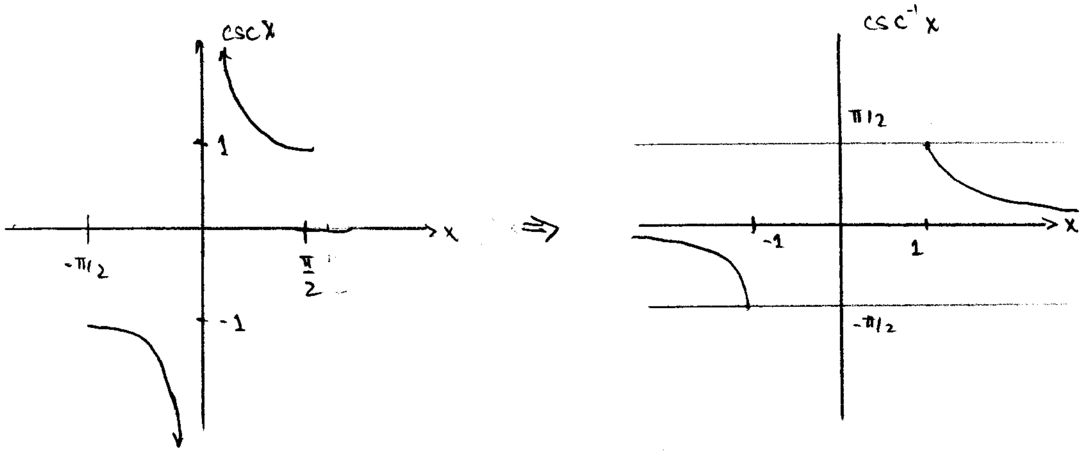
$$\sec\theta = \frac{x}{1} \quad \Rightarrow \quad \theta = \sec^{-1}x$$

$$\Rightarrow \quad \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

The graphs of the other inverse functions are:



Some useful identities:

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

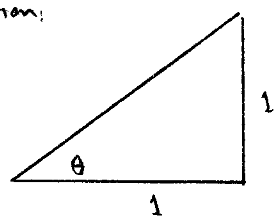
$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

<u>Example</u>

Find $\tan^{-1} 1$

Solution:




$\theta = \tan^{-1} 1$

$\Rightarrow \tan 1 = \theta$

Clearly $\theta = \frac{\pi}{4}$

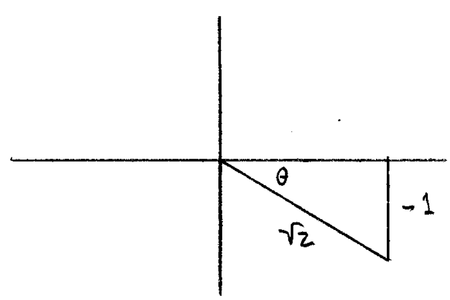
<u>Example</u>

Find $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:

$\theta = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \sin\theta = -\frac{1}{\sqrt{2}}$




The range of $\sin^{-1}x$ is $-\pi/2$ to $\pi/2$.
So $\sin^{-1}x$ for $x<0$ will imply a negative angle.

$\Rightarrow \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

More generally, note the even and oddness of the inverse trig functions:

$$\boxed{\begin{array}{lll} \sin^{-1}(-x) = -\sin^{-1}x & & \text{odd} \\ \cos^{-1}(-x) = \cos^{-1}x & & \text{even} \\ \tan^{-1}(-x) = -\tan^{-1}x & & \text{odd} \end{array}}$$

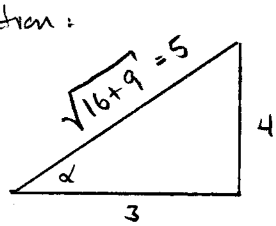
The inverse functions have the same even/odd nature as the corresponding trig functions. Another set of useful relationships:

$$\boxed{\begin{array}{l} \cot^{-1}(-x) = \pi - \cot^{-1}x \\ \sec^{-1}(-x) = \pi - \sec^{-1}x \\ \csc^{-1}(-x) = -\csc^{-1}(x) \end{array}}$$

<u>Example</u>

(#14, see 6.8) Given $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, $\cot\alpha$

Solution:




$\tan\alpha = \frac{4}{3}$

Hence

$\sin\alpha = \frac{4}{5}$

$\cos\alpha = \frac{3}{5}$

$\sec\alpha = \frac{5}{3}$

$\csc\alpha = \frac{5}{4}$

$\cot\alpha = \frac{3}{4}$

## Example

(# 22, sec 6.8) Find $\tan(\sec^{-1} 1) + \sin(\csc^{-1}(-2))$

Solution:

$\alpha = \sec^{-1} 1$

$\Rightarrow \sec\alpha = 1$

$\Rightarrow \cos\alpha = 1$

$\Rightarrow \alpha = 0$

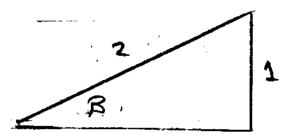
$\Rightarrow \tan\alpha = \tan 0 = 0$

$\sin(\csc^{-1}(-2)) = \sin(-\csc^{-1} 2)$

$= -\sin(\csc^{-1} 2)$

Now: $\beta = \csc^{-1} 2$

$\Rightarrow \csc\beta = 2$




$\Rightarrow \sin\beta = \frac{1}{2}$

$\Rightarrow -\sin\beta = -\frac{1}{2}$

So: $\tan(\sec^{-1} 1) + \sin(\csc^{-1}(-2)) = 0 - \frac{1}{2} = -\frac{1}{2}$

Note we could also have evaluated $\sin(\csc^{-1}(-2))$

$\sin(\csc^{-1}(-2)) = \sin(-\csc^{-1} 2)$

$= -\sin(\csc^{-1} 2)$

$= -\sin(\sin^{-1}(\frac{1}{2}))$ since $\csc^{-1}x = \sin^{-1}(\frac{1}{x})$

$= -\frac{1}{2}$ since $\sin x$ and $\sin^{-1}x$ are inverses.