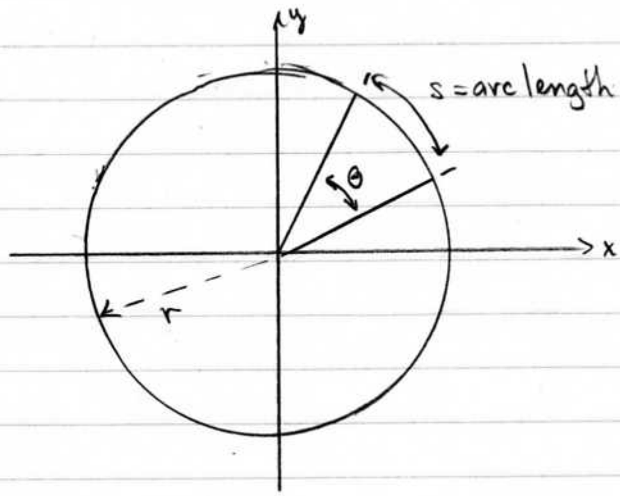
APPM 1350: Section P5: Trigonometric Functions

Angle Measurements

In everyday life we tend to think of angles as represented by degrees, for example degrees on a compass. A circle then has 360 degrees. And on a compass east would be 90 degrees, south 180 degrees, west 270 degrees, and north either 0 or 360 degrees.

However, in mathematics a more natural unit for an angle is the radian. The radian is motivated by the following idea. Consider a circle of radius r.



Consider some angle θ as shown. If you measure the distance along the circle in subtending this angle you get the arc length. The arc length is related to the radius by the useful equation

$$\boxed{s = r\theta} \qquad \theta \text{ in radians}$$

Now, we also know that the circumference is related to the radius by the equation

$$\text{circumference} = 2\pi r$$

But the circumference is just the arc length when θ goes all the way around the circle (i.e, $360°$). So if we let $s = \text{circumference} = r\theta_{360°}$ and plug this into the equation above we get

$$\cancel{r}\theta_{360°} = 2\pi \cancel{r}$$

$$\Rightarrow \theta_{360°} = 2\pi$$

So the angle θ associated with going all the way around the circle is 2π. Thus 2π radians is equivalent to 360 degrees. Hence we have the conversion between the two units

$$\boxed{2\pi \text{ radians} = 360 \text{ degrees}}$$

and more generally

$$\boxed{\begin{gathered} \theta_{\text{radians}} = \frac{2\pi}{360}\,\theta_{\text{degrees}} \\ \text{and} \\ \theta_{\text{degrees}} = \frac{360}{2\pi}\,\theta_{\text{radians}} \end{gathered}}$$

Example

Convert 270° to radians.

Solution:

$$\theta_{rad} = \frac{2\pi}{360}(270)$$

$$= 2\pi\left(\frac{270}{360}\right)$$

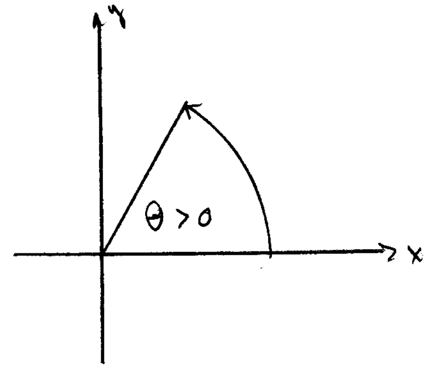
$$= 2\pi\left(\frac{3}{4}\right)$$

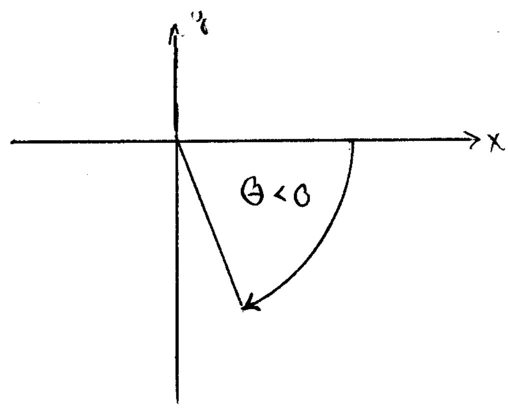
$$= \frac{3\pi}{2} \text{ radians}$$

Note the following equivalencies:

θ_{deg}	θ_{rads}
0	0
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π

An important point is that angles can take on any real value. Positive angles are associated with sweeping counterclockwise around the origin and negative angles are associated with sweeping clockwise.





But note that $\theta = -\frac{\pi}{2}$ points along the negative y axis. The angle $\theta = 3\pi/2$ also points along the negative y axis. So on a circle these angles end up at the same point.

What about $\theta = \frac{5}{2}\pi$? Note that

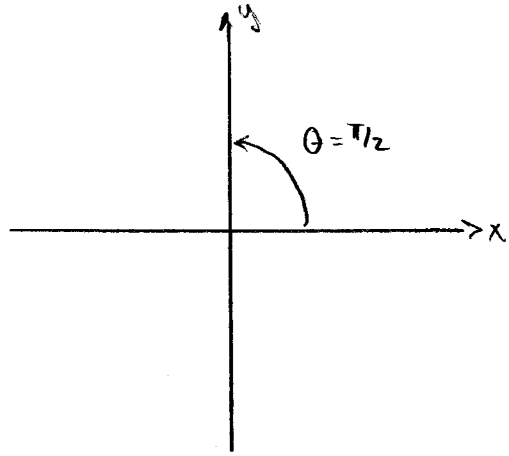
$$\theta = \frac{5}{2}\pi$$

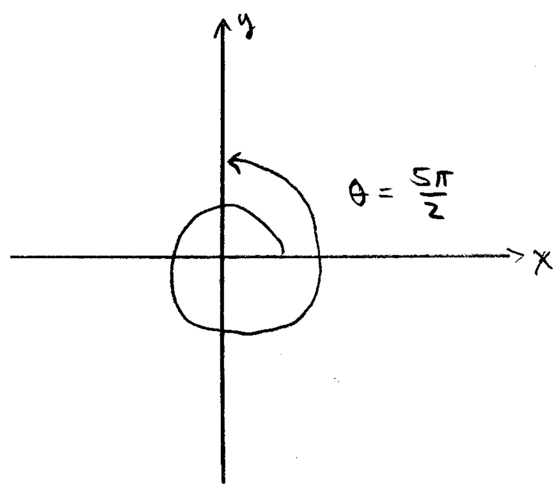
$$= \frac{4}{2}\pi + \frac{1}{2}\pi$$

$$= 2\pi + \frac{\pi}{2}$$

This is equivalent to having gone all the way around the circle and then an additron $\pi/2$ radians. So on a circle the angle $5/2\,\pi$

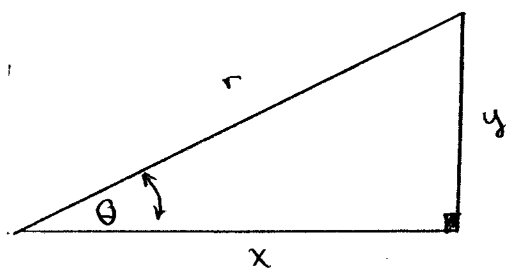
ends up at the same location as the angle $\pi/2$. But they aren't the same angle, to get to $\pi/2$ from $\theta = 0$ we moved $\pi/2$ radians counterclockwise from the positive x axis. But to get to $\frac{5\pi}{2}$ radians we moved a full 2π radians to go one complete revolution and then an additional $\pi/2$ radians. So while we end up at the same point on the circle, these are not the same angle because we got there in different ways.





Trigonometric Functions

Trigonometric functions are motivated by right triangles



Consider the right-triangle above. Relative to the vertex on the left that has an angle θ the length of the adjacent side of the triangle is x, the length of the opposite side is y, and the radius is r. We then define:

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

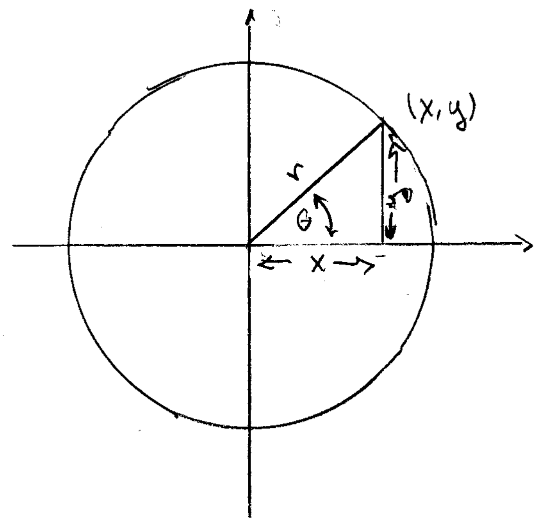
$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

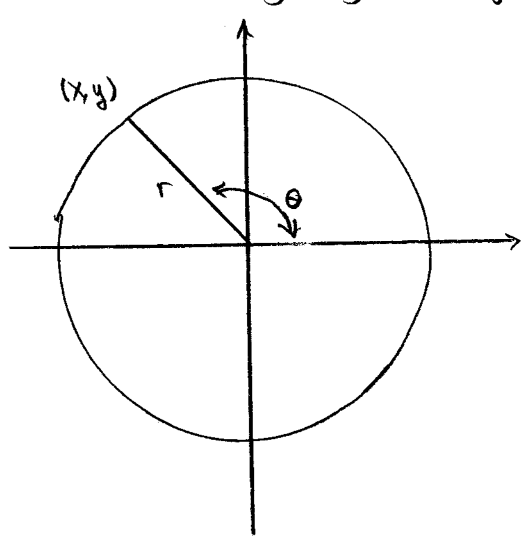
$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\cot = \frac{1}{\tan\theta} = \frac{x}{y}$$

For right triangles $\theta \in [0, \pi/2)$. But now consider a circle of radius r:



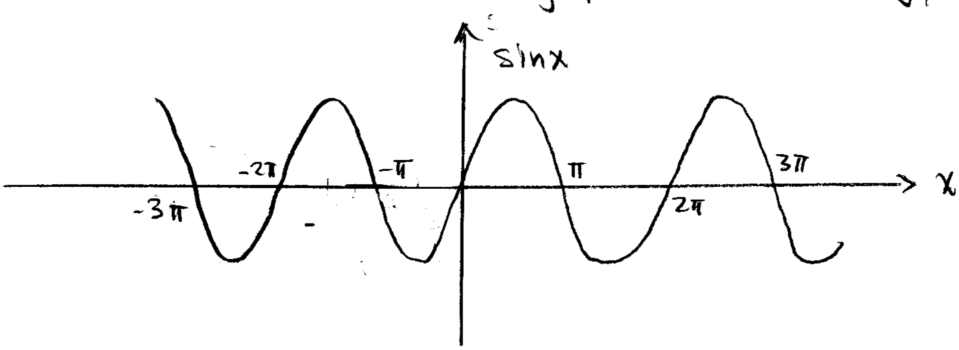
For an angle θ let us inscribe a right triangle of angle θ. The hypoteneuse starts at the vertex $(0,0)$ and extends to the vertex at a point (x,y). If $\theta \in [0, \pi/2)$ then this inscribed right-triangle looks like the one on the previous page and we can use the definitions of the trig functions on that same page. But if $\theta > \pi/2$ we no longer get a right triangle:



However, given any point (x,y) at an angle θ on this circle we can still define the trig functions using the definitions on page 6, <u>provided we do not divide by zero</u>. So in this way we can extend the trig functions to <u>any</u> angle $\theta \in \mathbb{R}$ except for points where there is a division by zero. Thus:

Definition	Domain
$\sin\theta = \frac{y}{r}$	$D = \mathbb{R}$
$\cos\theta = \frac{x}{r}$	$D = \mathbb{R}$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$D = \{\theta \in \mathbb{R} : \theta \neq \pm n\frac{\pi}{2},\ n = 1,3,5,\ldots\}$
$\csc\theta = \frac{1}{\sin\theta}$	$D = \{\theta \in \mathbb{R} : \theta \neq \pm n\pi,\ n = 0,1,2,\ldots\}$
$\sec\theta = \frac{1}{\cos\theta}$	$D = \{\theta \in \mathbb{R} : \theta \neq \pm n\frac{\pi}{2},\ n = 1,3,5,\ldots\}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$D = \{\theta \in \mathbb{R} : \theta \neq \pm n\pi,\ n = 0,1,2,\ldots\}$

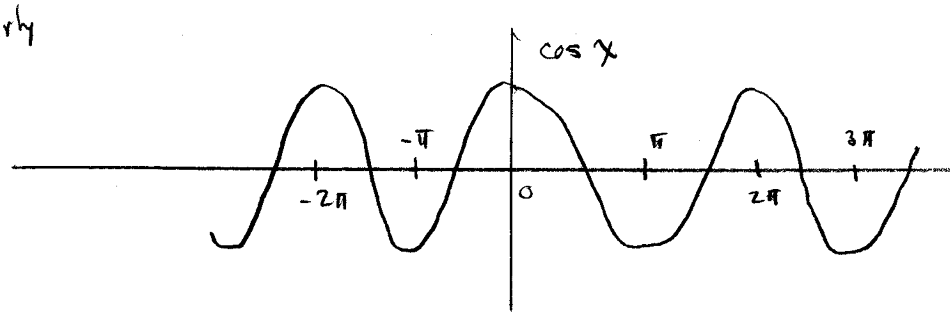
So we get the numerical value of a trig function at an angle θ by looking at the ratios of $x, y,$ and r on a circle of radius r. In this way we define those values for arbitrary angles θ. Put slightly differently then, we can consider the trig functions as just functions on the real line and graph them accordingly. So



$D = \mathbb{R}$
$R = [-1,$

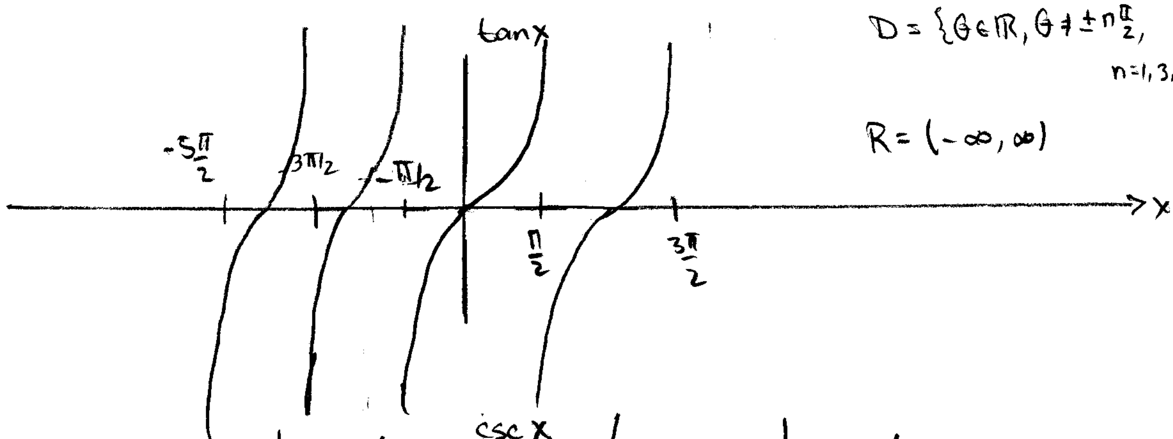
is the graph of $\sin x$ where x now is just some real number.

Similiarly



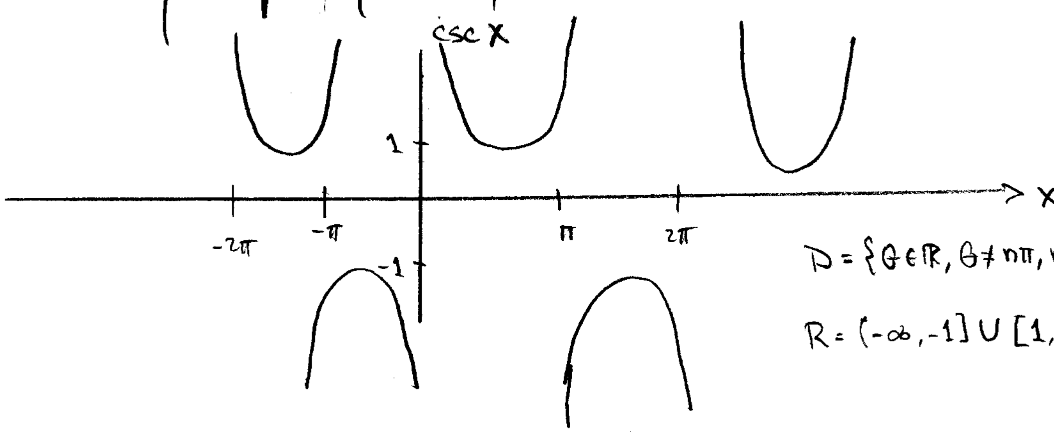
$D = \mathbb{R}$

$R = [-1, 1]$



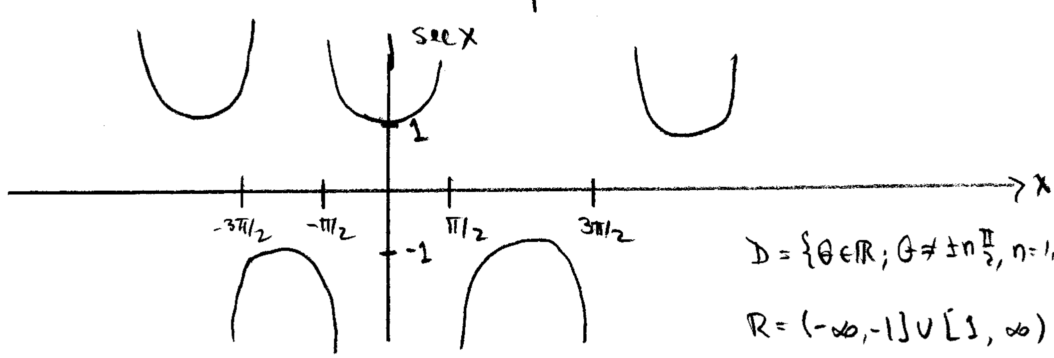
$D = \{\theta \in \mathbb{R}, \theta \neq \pm n\frac{\pi}{2}, n = 1, 3, 5\}$

$R = (-\infty, \infty)$



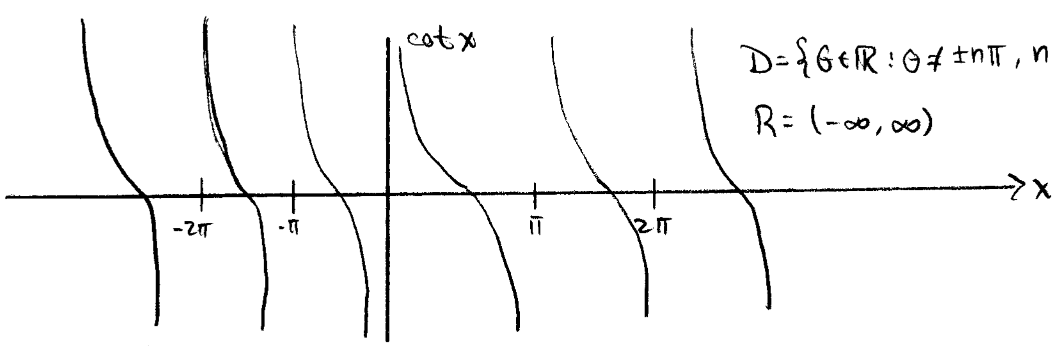
$D = \{\theta \in \mathbb{R}, \theta \neq n\pi, n \in \mathbb{Z}\}$

$R = (-\infty, -1] \cup [1, \infty)$



$D = \{\theta \in \mathbb{R}; \theta \neq \pm n\frac{\pi}{2}, n = 1, 3, 5 ..\}$

$R = (-\infty, -1] \cup [1, \infty)$



$D = \{\theta \in \mathbb{R} : \theta \neq \pm n\pi, n \in \mathbb{Z}\}$

$R = (-\infty, \infty)$

Periodicities

The striking thing about these graphs is that they repeat themselves. So for example

$$\sin x = \sin(x + 2\pi)$$

or

$$\sin x = \sin(x - 2\pi)$$

The value of the function is the same if we add (or subtract) 2π to the value of x. This is an example of <u>periodicity</u>.

> Def: A function $f(x)$ is <u>periodic</u> if there is a number $p > 0$ such that $f(x+p) = f(x)$ for all x. The smallest p for which this is true is the <u>period</u> of f.

> So $\sin x$ and $\cos x$ are periodic with period 2π. All the other trig functions are periodic with period π.

Even and Odd

Recall that $f(x)$ is <u>even</u> if $f(x) = f(-x)$ and is <u>odd</u> if $f(x) = -f(-x)$. The trig functions are all even or odd:

Even	Odd
$\cos(x) = \cos(-x)$	$\sin x = -\sin(-x)$
$\sec(x) = \sec(-x)$	$\tan x = -\tan(-x)$
	$\csc x = -\csc(-x)$
	$\cot x = -\cot(-x)$

Trigonometric Identities

There are many identities that trig functions obey. If you remember only one of them, memorize this one!

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Two other very important ones are:

$$\boxed{\begin{aligned} 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$

Why is $\sin^2\theta + \cos^2\theta = 1$ the more important to remember? Because you can derive the other two from that one.

Example

Prove that $1 + \tan^2\theta = \sec^2\theta$

Proof:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{\cos^2\theta}\left[\sin^2\theta + \cos^2\theta\right] = \frac{1}{\cos^2\theta}[1]$$

$$\Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + 1 = \sec^2\theta$$

$$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

$\blacksquare$

Example

Prove that $1 + \cot^2\theta = \csc^2\theta$

Proof:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{\sin^2\theta}\left[\sin^2\theta + \cos^2\theta\right] = \frac{1}{\sin^2\theta}[1]$$

$$\Rightarrow 1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow 1 + \cot^2\theta = \csc^2\theta$$

∎

Other useful identities:

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\sin(A+B) = \sin A\cos B + \cos A\sin B$	Angle sum formulas
$\cos 2A = \cos^2 A - \sin^2 A$ $\sin 2A = 2\sin A\cos A$	Double-angle formulas
$\cos^2 A = \dfrac{1+\cos 2A}{2}$ $\sin^2 A = \dfrac{1-\cos 2A}{2}$	More double-angle formulas

Memorize the three identities on the previous page. Don't bother memorizing these other identities, you can look these up in a book or in a pinch derive them.