Answerkey Exam #2: APPM 1350 - Spring 2005.
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P1. (a) $f'(x) = (2x - 2x^{-3})\tan(x) + (x^2 + 1 + x^{-2})\sec^2(x)$.

(b) $g'(t) = \dfrac{2(t^2-1) - (2t+1)2t}{(t^2-1)^2} = \ldots = -\dfrac{2+2t+2t^2}{(t^2-1)^2}$.

(c) $2 \cdot \dfrac{1}{4} y^{-3/4} \dfrac{dy}{dx} = 2x - \dfrac{dy}{dx} \Rightarrow \left(1 + \dfrac{y^{-3/4}}{2}\right)\dfrac{dy}{dx} = 2x \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{1 + \frac{y^{-3/4}}{2}} = \dfrac{4x}{2 + y^{-3/4}}$.

(d) $h'(t) = -\sin\left(\sqrt{1+\sqrt{t}}\right) \cdot \dfrac{1}{2}(1+\sqrt{t})^{-1/2} \cdot \dfrac{1}{2} t^{-1/2} = -\dfrac{\sin\left(\sqrt{1+\sqrt{t}}\right)}{4\sqrt{t}\sqrt{1+\sqrt{t}}}$.

(e) $\displaystyle \lim_{x\to\infty} \frac{x - \sin(2x)}{x + \sin(3x)} = \lim_{x\to\infty} \frac{x\left(1 - \frac{\sin(2x)}{x}\right)}{x\left(1 - \frac{\sin(3x)}{x}\right)} = \lim_{x\to\infty} \frac{1 - \frac{\sin(2x)}{x}}{1 - \frac{\sin(3x)}{x}} = \frac{1-0}{1+0} = 1,$

because (according to the sandwich theorem) the $\lim_{x\to\infty} \frac{\sin(2x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\sin(3x)}{x} = 0$.

(f) Like above we have that $\displaystyle \lim_{t\to 0} \frac{t - \sin(2t)}{t + \sin(3t)} = \lim_{t\to 0} \frac{1 - \frac{\sin(2t)}{t}}{1 - \frac{\sin(3t)}{t}}$.

On the other hand, since the $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, then

$$\lim_{t\to 0} \frac{\sin(2t)}{t} = 2 \lim_{t\to 0} \frac{\sin(2t)}{2t} = 2 \lim_{x\to 0} \frac{\sin(x)}{x} = 2.$$

A similar argument shows that

$$\lim_{t\to 0} \frac{\sin(3t)}{t} = 3.$$

As a result,

$$\lim_{t\to 0} \frac{t - \sin(2t)}{t + \sin(3t)} = \frac{1-2}{1+3} = -\frac{1}{4}.$$

P2. To have $f(x)$ differentiable at $x = \pi$ we must have first $f(x)$ continuous at $x = \pi$. Continuity from the right is guarantied. To have continuity from the left we need

$$f(\pi) = \lim_{x\to\pi^-} f(x) \Leftrightarrow \sin(\pi) = a\pi + b \Leftrightarrow a\pi + b = 0\,.$$

To have $f(x)$ now differentiable at $x = \pi$ we need the left and right derivatives to be equal at $x = \pi$ i.e. we need

$$\cos(\pi) = a \Leftrightarrow a = -1\,.$$

Since $a\pi + b = 0$ then $b = \pi$.

P3. (a) Let x the distance between the bottom of the ladder and the house. Let y be the hight of the top of the ladder. Because the ladder is 15 ft long, and the top of the ladder is at all times in contact with the house, using Pythagoras' theorem we determine that:

$$x^2 + y^2 = 15^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}\,.$$

In our case, when $x = 12$, $\frac{dx}{dt} = 3$ and $y = \sqrt{15^2 - 12^2} = 9$. Therefore,

$$\frac{dy}{dt} = -4 \text{ ft/sec},$$

i.e. the top of the ladder is approaching the ground at a speed of 4 ft/sec.

(b) Call θ the angle of inclination between the ladder and the ground. At all times, it applies that

$$\cos(\theta) = \frac{x}{15} \Rightarrow -\sin(\theta)\frac{d\theta}{dt} = \frac{1}{15}\frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{15\sin(\theta)}\frac{dx}{dt}.$$

Since $15\sin(\theta) = y$ at all times, we obtain that

$$\frac{d\theta}{dt} = -\frac{1}{y}\frac{dx}{dt}.$$

In particular, since $\frac{dx}{dt} = 3$ and $y = 9$ when $x = 12$, we obtain that

$$\frac{d\theta}{dt} = -\frac{1}{9} \cdot 3 = -\frac{1}{3} \text{ rad/sec}.$$

P4. (a) $f(-x) = \frac{(-x)^3+1}{(-x)^2} = \frac{1-x^3}{x^2}$. Thus, except for very specific values of x, in general we have that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

(b) Observe that $f(x) = x + \frac{1}{x^2}$. Therefore $\lim_{x\to\pm\infty}(f(x) - x) = \lim_{x\to\pm\infty}\frac{1}{x^2} = 0$. Hence $y = x$ is an oblique asymptote.

(c) No by part (b).

(d) Since $\lim_{x\to 0} f(x) = +\infty$, $x = 0$ is a vertical asymptote. (Since f is continuous elsewhere there are no other vertical asymptotes).

(e) $f'(x) = \frac{d}{dx}\left(x + \frac{1}{x^2}\right) = 1 - \frac{2}{x^3} = \frac{x^3-2}{x^3}$.

(f) We first look for solutions of the equation: $f'(x) = 0$, $x \neq 0$. Using (e) one finds that $x = 2^{1/3}$ is the only critical point in the domain. To determine where f is increasing/decreasing we study the sign of f' on the intervals

	$(-\infty, 0)$	$(0, 2^{1/3})$	$(2^{1/3}, +\infty)$
$x^3 - 2$	$-$	$-$	$+$
x^3	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

Therefore, $f(x)$ is increasing on the interval $(-\infty, 0)$ and $(2^{1/3}, +\infty)$.

(g) Based on the table in part (f), we conclude that $f(x)$ is decreasing on the interval $(0, 2^{1/3})$.

(h) $f''(x) = \frac{d}{dx}\left(1 - \frac{2}{x^3}\right) = \frac{6}{x^4}$.

(i) Observe that $f'(x) > 0$, for all $x \neq 0$. Thus $f(x)$ is concave up on $(-\infty, 0)$ and on $(0, +\infty)$. (It is not right to say that f is concave up on $(-\infty, \infty)$ because f is not defined at $x = 0$.)

(j) Based on (i), f is concave down nowhere.

(k) There can be no cusp in the graph of $y = f(x)$ because $f(x)$ is differentiable at all points in its domain.

Based on the table in part (f), it follows that f has a local minimum at $x = 2^{1/3}$. (However, this is not an absolute minimum. This is because the $\lim_{x\to-\infty} f(x) = -\infty$.)