INSTRUCTIONS: Books, crib sheets and electronic devices are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

---

1. (15 points) Answer the following questions as either **ALWAYS TRUE** or **NOT ALWAYS TRUE**. For this problem only, you do not need to justify your answer.

   (a) If $f(x)$ is continuous at x_0, then it is differentiable at x_0. NOT ALWAYS TRUE

   (b) If $|2x-3|>1$, then $1<x<2$. NOT ALWAYS TRUE

   (c) If $f(x)$ is an odd function and $\lim_{x\to a^+} f(x)=L$, then $\lim_{x\to -a^-} f(x)=-L$ ALWAYS TRUE

   (d) If $p(x)$ is a polynomial, then $\lim_{x\to 2^+}\dfrac{p(x)}{x-2}=+\infty$. NOT ALWAYS TRUE

   (e) If $f'(x)=g'(x)$, then $f(x)=g(x)$. NOT ALWAYS TRUE

2. (25 points) Compute the following limits, if they exist. If the limit does not exist, clearly state this.

   (a) $\lim_{x\to 0} x^4\sin\left(\dfrac{1}{x}\right)$. Note that $\lim_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ DNE. We know that $-1\le \sin\dfrac{1}{x}\le 1$ so then $-x^4\le x^4\sin\dfrac{1}{x}\le x^4$. Since $\lim_{x\to 0}-x^4=\lim_{x\to 0}x^4=0$, then by the Sandwich Thm. $\lim_{x\to 0} x^4\sin\left(\dfrac{1}{x}\right)=0$.

   (b) $\lim_{x\to -1}\dfrac{x^2-4x-5}{x^2-1}$ is indeterminate. But $\lim_{x\to -1}\dfrac{x^2-4x-5}{x^2-1}=\lim_{x\to -1}\dfrac{(x-5)(x+1)}{(x-1)(x+1)}=\lim_{x\to -1}\dfrac{(x-5)}{(x-1)}=\dfrac{-6}{-2}=3$.

   (c) $\lim_{x\to 2^-}\dfrac{x-3}{x^2-4}=\dfrac{-1^-}{0^-}=+\infty$.

   (d) $\lim_{x\to 4}\dfrac{\frac{1}{x}-\frac{1}{4}}{x-4}$ is indterminate. Put numerator over a common denominator to arrive at $\lim_{x\to 4}\dfrac{4-x}{4x(x-4)}=\lim_{x\to 4}\dfrac{-1}{4x}=\dfrac{-1}{16}$.

   (e) $\lim_{x\to 3} f(x)$ where $f(x)=\begin{cases}2x^2-6x+5, & \text{if } x\le 3\\ x^3-x^2-3x, & \text{if } x>3\end{cases}$. Look at limits from the left and right: $\lim_{x\to 3^-} 2x^2-6x+5=18-18+5=5$ and $\lim_{x\to 3^+} x^3-x^2-3x=27-9-9=27$. Since the left and right hand limits differ, the limit DNE.

3. (25 points) Find the requested information.

   (a) State the definition of the derivative, $\dfrac{df}{dx}$. The definition is $\dfrac{df}{dx}=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

   (b) Using this definition, calculate the derivative with respect to x of $f(x)=x^2+x$.
   $f'(x)=\lim_{h\to 0}\dfrac{(x+h)^2+(x+h)-(x^2+x)}{h}=\lim_{h\to 0}\dfrac{2xh+h^2+h}{h}=\lim_{h\to 0}2x+h+1=2x+1$.

(c) Detemine the equation of the tangent line to $f(x)$ at $x = 2$. The tangent line passes through the point $(2, f(2)) = (2, 6)$ with the slope $m = f'(2) = 2 \cdot 2 + 1 = 5$ Using the point-slope form of a straight line, $y_{\text{tangent}} = 6 + 5(x - 2)$.

(d) Detemine the equation of the normal line to $f(x)$ at $x = 2$. The normal passes through the same point $(2, 6)$ but with slope $m_{\text{normal}} = -1/m_{\text{tangent}} = -1/5$. Again, using the point–slope form, $y_{\text{normal}} = 6 + \dfrac{-1}{5}(x - 2)$.

4. (20 points) Using appropriate rules of differentiation, evaluate the following derivatives. For parts (a) and (b) do not simplify your work beyond the point of having no derivatives remaining in your result.

(a) $\dfrac{d}{dx}\left(3x^4 + \dfrac{7}{x^2} + \pi^2\right) = 12x^3 - 14x^{-3} + 0.$

(b) $f'(x)$ where $f(x) = \dfrac{x^2+3}{x(x^4+1)}$. We need
$$\frac{d}{dx}\frac{x^2+3}{(x^5+x)} = \frac{(x^2+3)'(x^5+x) - (x^2+3)(x^5+x)'}{(x^5+x)^2} = \frac{(2x)(x^5+x) - (x^2+3)(5x^4+1)}{(x^5+x)^2}$$

(c) $f'(2)$ if $f(x) = \dfrac{u(x)\,v(x)}{w(x)}$ where $u(2) = 3$, $v(2) = 4$, $w(2) = 2$, and $u'(2) = 1$, $v'(2) = 5$ and $w'(2) = 5$. We need $\dfrac{d}{dx}\dfrac{u(x)\,v(x)}{w(x)} = \dfrac{(u\,v)'w - (u\,v)w'}{w^2} = \dfrac{(u'\,v + u\,v')w - (u\,v)w'}{w^2}$.
Evaluating this at $x = 2$ yields $f'(2) = \dfrac{(u'(2)v(2) + u(2)v'(2))w(2) - u(2)v(2)w'(2)}{w^2(2)} =$
$\dfrac{(1 \cdot 4 + 3 \cdot 5)2 - 3 \cdot 4 \cdot 5}{2^2} = -11/2.$

5. (15 points) Are there numbers a and b that will make the function
$$f(x) = \begin{cases} ax^2 + 3b & \text{for } x < 1 \\ ax^4 - 3x + 4b & \text{for } x \geq 1 \end{cases}$$
differentiable at $x = 1$? If yes, determine their values. Be sure to show your work and clearly justify your answers.

We require that $f(x)$ be continuous at $x = 1$, so setting $\lim_{x\to 1^-} ax^2 + 3b = \lim_{x\to 1^+} ax^4 - 3x + 4b$ leads to $a + 3b = a - 3 + 4b$. Thus $b = 3$. We also require that each of these limits equal $f(1)$, but we can't verify this until we have a value for a.

We need the 1–sided derivatives from the left and right have the same value, so we set $\lim_{x\to 1^-} 2ax = \lim_{x\to 1^+} 4ax^3 - 3$ which leads to $2a = 4a - 3$, or $a = 3/2$.

Now that we have a value for a, we can go back and check the continuity of $f(x)$ at $x = 1$. In particular, evaluating the expressions (from above) $a + 3b = a - 3 + 4b$ with $a = 3/2$ and $b = 3$ we get $21/2$ on each side. In other words, we now have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 21/2$. Hence we can say $\lim_{x\to 1} f(x) = 21/2$. Lastly we need to verify that $f(1)$ equals the value of the limit. From the definition of the function we see that $f(1) = (ax^4 - 3x + 4b)|_{x=1} = \dfrac{3}{2} - 3 + 4 \cdot 3 = 21/2.$