APPM 1350 **EXAM 2** **FALL 2005**

INSTRUCTIONS: Books, crib sheets and electronic devices are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and $\boxed{\text{box}}$ your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (15 points) Answer the following questions as either **ALWAYS TRUE** or **NOT ALWAYS TRUE**. For this problem only, you do not need to justify your answer.

   (a) The derivative of a (differentiable) even function is an odd function. We threw out this problem because it was missing the word "differentiable."

   (b) The point $x = 0$ is an inflection point of the function $y = x^4 - 2x + 6$. NOT ALWAYS TRUE. The point $x = 0$ is not the location of an inflection point.

   (c) If $f(x)$ is differentiable everywhere and $f(3) = f(6) = 0$, then $f(x)$ has a critical point in $(3, 6)$. ALWAYS TRUE. This is Rolle's Thm.

   (d) If $f(x)$ and $g(x)$ are increasing functions on an interval I, then the product $f(x)\,g(x)$ is increasing on I. NOT ALWAYS TRUE. A counter example can be constructed using the functions $f(x) = x$ and $g(x) = x^3$ on the interval $(-2, -1)$.

   (e) The graph of $y = \dfrac{4x^{2005} + 30x^{1776} - x}{7x^{2005} + 5x^{1215}}$ has a horizontal asymptote. ALWAYS TRUE. The horizontal asymptote is $y = 4/7$.

2. (30 points) Compute the following. (If a limit does not exist, clearly state this.)

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{\tan(3x)} = \lim_{x\to 0} \frac{\sin(2x)}{2x}\frac{3x}{\sin(3x)}\cos(3x)\frac{2x}{3x} = 1 \cdot 1 \cdot 1 \cdot \frac{2}{3} = \frac{2}{3}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{x^2 + 3x - \sin x}{4 + 3x^2 + \cos(x^2)} = \lim_{x\to\infty} \frac{1 + \frac{3}{x} - \frac{\sin(x)}{x^2}}{\frac{4}{x^2} + 3 + \frac{\cos(x^2)}{x^2}} = \frac{1 + 0 - 0}{0 + 3 + 0} = \frac{1}{3}$ where the last limit in both the numerator and denominator is zero by the Sandwich Thm.

   (c) $\displaystyle\lim_{x\to\infty} \frac{x^{-1} + 3x^{-2}}{4x^{-3} + 8x^{-1}} = \lim_{x\to\infty} \frac{1 + 3x^{-1}}{4x^{-2} + 8} = \frac{1}{8}.$

   (d) $\dfrac{dy}{dx}$ if $\sec^2(xy) = x^2 + y^3$. Using implicit differentation, $\dfrac{dy}{dx} = \dfrac{2x - 2y\sec^2(xy)\tan(xy)}{2x\sec^2(xy)\tan(xy) - 3y^2}$.

   (e) $\dfrac{dy}{dx}$ if $y = (2x-3)^5(x^2+5)^7$. Then
   $$\frac{dy}{dx} = 5(2x-3)^4 2(x^2+5)^7 + (2x-3)^5 7(x^2+5)^6 2x = 10(2x-3)^4(x^2+5)^7 + 14x(2x-3)^5(x^2+5)^6$$

3. (20 points) The thin lens equation in optics is

   $$\frac{1}{D} + \frac{1}{S} = \frac{1}{F},$$

   where D is the distance from the object to the lens, S is the distance from the image to the lens, and F, a constant, is the focal length of the lens. Suppose that a particular lens has a focal length of $F = 6$ cm and that an object is moving towards the lens at the rate of 2 cm/s. How fast is the image distance S changing at the moment when the distance between the object and the lens is $D = 10$ cm?

   First note that at the instant in question, one can solve for S from $\dfrac{1}{S} = \dfrac{1}{F} - \dfrac{1}{D}$. Since $F = 6$ cm and $D = 10$ cm, then $S = 15$ cm.

Now, calculating $\frac{d}{dt}\left(\frac{1}{D}+\frac{1}{S}\right) = \frac{d}{dt}\left(\frac{1}{F}\right)$ we get $-\frac{\dot{D}}{D^2} - \frac{\dot{S}}{S^2} = 0$ from which one can solve for $\dot{S} = -\dot{D}\frac{S^2}{D^2} = -\left(-2\frac{\text{cm}}{\text{sec}}\right)\left(\frac{15 \text{ cm}}{10 \text{ cm}}\right)^2 = \frac{9}{2}$ cm/sec.

4. (20 points) You want to build a pool in your backyard for your prize–winning Koi. The pool should have a square bottom and hold V gallons of water. You want to put decorative tile on the vertical sides of the pool, but to save money, the bottom will be plain concrete. The tile costs four times as much per square foot as the concrete. What pool dimensions will minimize the cost?

   Let the length a side of the bottom of the pool be x and the depth of the pool be y. Then the volume of the pool (which is fixed) is $V = x^2y$. The cost of the pool, C, is then $C = x^2 + 4(4xy)$. We can use the constraint ($V = x^2y$) to eliminate one of the variables, in this case, $y = V/x^2$. Substituting this expression for y into the cost equation yields $C(x) = x^2 + 16\frac{V}{x}$. We need to find the minimum of C on the interval $0 < x < \infty$.

   We have no end points, so we only need to investigate critical points. Calculating $C' = 2x - \frac{16V}{x^2}$, we see that there are no values of x in the interval of interest that make C' undefined. However setting $C' = 0$ indicates that $x = 2\sqrt[3]{V}$ is the only critical point of interest. We need to classify the extreme value associated with our critial point, so we calculate $C'' = 2 + 32V/x^3$. For values of x in the interval of interest we find that $C'' > 0$ indicating that our critical point $x = 2\sqrt[3]{V}$ is a minimum.

   Finally, to find the corresponding value of y, we use the constraint to calculate $y = V/x^2 = V/(2\sqrt[3]{V})^2 = \sqrt[3]{V}/4$. Hence, the dimensions are $x = 2\sqrt[3]{V}$ and $y = \sqrt[3]{V}/4$.

5. (15 points) A twice–differentiable function $y = f(x)$ has an x–intercept at $x = 4$. In addition, the signs of the first and second derivatives in the given intervals are as follows:

	$(-\infty, 0)$	$(0, 2)$	$(2, 4)$	$(4, 6)$	$(6, \infty)$
$f'(x)$	+	−	−	+	+
$f''(x)$	−	−	+	+	−

   Using all of the above information, carefully sketch the graph of $y = f(x)$, labeling all maxima, minima and inflection points.