1. (15 points) For each of the following unrelated questions, answer either ALWAYS TRUE, ALWAYS FALSE or NEITHER. No justification is necessary.

   (a) If $f(x) > 1$ for all x and $\lim\limits_{x\to 0} f(x)$ exists, then $\lim\limits_{x\to 0} f(x) > 1$. (NEITHER)

   (b) All continuous functions have derivatives. (ALWAYS FALSE)

   (c) The derivative of the function $\tan^2 x$ is the derivative of $\sec^2 x$. (ALWAYS TRUE)

   (d) $\displaystyle\int_{-5}^{5} (ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$ (ALWAYS TRUE)

   (e) If $\lim_{x\to 6} f(x)g(x)$ exists, then the limit is $f(6)g(6)$. (NEITHER)

2. (21 points) Find $\frac{dy}{dx}$ in each case. No simplification is necessary.

   (a) $y = x\ln(\arccos x)$

$$\begin{aligned}\frac{dy}{dx} &= (x)\left(\frac{1}{\arccos x}\right)\left(\frac{-1}{\sqrt{1-x^2}}\right) + (1)(\ln(\arccos x))\\ &= \frac{-x}{\arccos x\sqrt{1-x^2}} + \ln(\arccos x)\end{aligned}$$

   (b) $y = x^{e^x}$

$$\begin{aligned}\ln y &= e^x \ln x\\ \Rightarrow \frac{1}{y}\frac{dy}{dx} &= (e^x)\left(\frac{1}{x}\right) + (e^x)(\ln x)\\ \Rightarrow \frac{dy}{dx} &= y\,e^x\left(\frac{1}{x} + \ln x\right)\\ &= x^{e^x} e^x\left(\frac{1}{x} + \ln x\right)\end{aligned}$$

   (c) $x\,e^y = \ln xy + \arctan y$

$$\begin{aligned}(x)\left(e^y \cdot \frac{dy}{dx}\right) + (1)(e^y) &= \frac{1}{xy}\left(x\frac{dy}{dx} + 1\cdot y\right) + \frac{1}{1+y^2}\cdot\frac{dy}{dx}\\ \frac{dy}{dx}\left(x\,e^y - \frac{1}{y} - \frac{1}{1+y^2}\right) &= \frac{1}{x} - e^y\\ \frac{dy}{dx} &= \frac{\frac{1}{x} - e^y}{xe^y - \frac{1}{y} - \frac{1}{1+y^2}}\end{aligned}$$

3. (28 points) Evaluate each of the following limits, if it exists. If the limit does not exist, state this and state your justification. Show all your work.

(a) $\lim_{t\to 0^+} t^{t^2}$

$$\begin{aligned}
\lim_{t\to 0^+} t^{t^2} &= \lim_{t\to 0^+} e^{t^2 \ln t} = e^{\lim_{t\to 0^+} t^2 \ln t} \\
\lim_{t\to 0^+} t^2 \ln t &= \lim_{t\to 0^+} \frac{\ln t}{t^{-2}} \overset{L'H}{=} \lim_{t\to 0^+} \frac{t^{-1}}{-2t^{-3}} = \lim_{t\to 0^+} -\frac{1}{2}t^2 = 0 \\
\Rightarrow \lim_{t\to 0^+} t^{t^2} &= e^0 = 1
\end{aligned}$$

(b) $\lim_{r\to\infty} \left(re^{1/r} - r\right)$

$$\begin{aligned}
\lim_{r\to\infty} \left(re^{1/r} - r\right) &= \lim_{r\to\infty} r\left(e^{1/r} - 1\right) = \lim_{r\to\infty} \frac{e^{1/r} - 1}{\frac{1}{r}} \\
&\overset{L'H}{=} \lim_{r\to\infty} \frac{e^{1/r} \cdot \frac{-1}{r^2}}{\frac{-1}{r^2}} = \lim_{r\to\infty} e^{1/r} = e^0 \\
&= 1
\end{aligned}$$

(c) $\lim_{x\to 0} x \operatorname{arccot} \frac{1}{x}$

$$\begin{aligned}
\lim_{x\to 0} x \operatorname{arccot} \frac{1}{x} &= \lim_{x\to 0} \frac{\operatorname{arccot} \frac{1}{x}}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{-1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to 0} \frac{-1}{1+\frac{1}{x^2}} = \lim_{x\to 0} \frac{-x^2}{x^2+1} \\
&= 0
\end{aligned}$$

(d) $\lim_{h\to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1+t^3}\, dt$

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1+t^3}\, dt &= \lim_{h\to 0} \frac{\int_a^{2+h} \sqrt{1+t^3}\, dt - \int_a^2 \sqrt{1+t^3}\, dt}{h} \\
&= F'(2) \text{ where } F(x) = \int_a^x \sqrt{1+t^3}\, dt \\
F'(x) &= \sqrt{1+t^3} \qquad \text{(By FTC pt 1)} \\
\Rightarrow F'(2) &= \sqrt{1+2^3} = \sqrt{9} = 3
\end{aligned}$$

4. (21 points) Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{e^{-x}}{1+e^{-2x}}\,dx$

$$u = e^{-x} \qquad du = -e^{-x}dx$$

$$\int \frac{e^{-x}}{1+e^{-2x}}\,dx = \int \frac{-1}{1+u^2}\,du = \text{arccot}u + C = \text{arccot}\big(e^{-x}\big) + C$$

(b) $\displaystyle\int \tan x \ln(\cos x)\,dx$

$$u = \ln(\cos x) \qquad du = \frac{1}{\cos x}\cdot(-\sin x)\,dx = -\tan x\,dx$$

$$\int \tan x \ln(\cos x)\,dx = \int -u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}(\ln\cos x)^2 + C$$

(c) $\displaystyle\int_0^2 \frac{t}{9-t^2}\,dt$

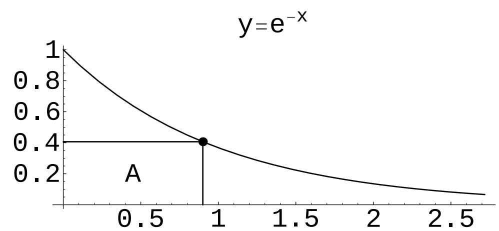
$$u = 9-t^2 \qquad du = -2t\,dt \qquad t\,dt = -\frac{1}{2}du$$

$$\int_0^2 \frac{t}{9-t^2}\,dt = \int_9^5 \frac{-1}{2u}\,du = -\frac{1}{2}\ln|u|\Big|_9^5 = -\frac{1}{2}\ln 5 + \frac{1}{2}\ln 9 = \frac{1}{2}\ln\frac{9}{5}$$

5. (25 points) Do ONE of the following two problems. State clearly which problem you have chosen in your bluebook. Only work for the chosen problem will be graded.

   (a) What is the area of the largest rectangle in the first quadrant with two sides on the axes and one vertex on the curve $y = e^{-x}$?

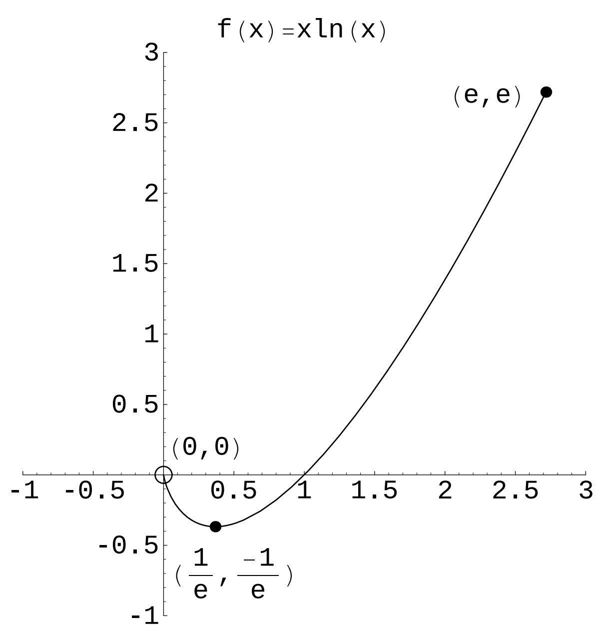
   Solution: First, draw a picture:



$$
\begin{aligned}
A &= (x)(y) = x\,e^{-x} \\
\Rightarrow \frac{dA}{dx} &= (x)(-e^{-x}) + (1)(e^{-x}) = e^{-x}(1-x) = 0 \\
&\Rightarrow x = 1 \\
&\Rightarrow A = 1 \cdot e^{-1} = \frac{1}{e}
\end{aligned}
$$

   (b) Plot the function $f(x) = x \ln x$ on $(0, e]$. Find and label all critical points, local and absolute extrema, and inflection points.

$$
\begin{aligned}
f'(x) &= 1 + \ln x = 0 \qquad \Rightarrow x = \frac{1}{e} \\
f''(x) &= \frac{1}{x} > 0 \text{ for all } x \text{ in } (0, e]
\end{aligned}
$$



6. (40 points)

   (a) What does it mean for $f(x)$ to be continuous at $x = a$?
   $\Rightarrow f$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

   (b) What does it mean for $f(x)$ to be differentiable at $x = a$?
   $\Rightarrow f$ is differentiable at $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists or, alternatively, $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

   (c) State both parts of the Fundamental Theorem of Calculus.
   FTC Part 1: If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then for $a \le x \le b$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

   FTC Part 2: If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

   (d) If f is a continuous function such that

$$\int_0^x f(t)\,dt = x \sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt$$

   for all x, find an explicit formula for $f(x)$.

$$\begin{aligned} \frac{d}{dx}\int_0^x f(t)\,dt &= \frac{d}{dx}\left(x \sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt\right) \\ f(x) &= x \cos x + \sin x + \frac{f(x)}{1+x^2} \\ \Rightarrow f(x) &= \frac{x \cos x + \sin x}{1 - \frac{1}{1+x^2}} \end{aligned}$$