1. (12 points)
   (a) Neither
   (b) Always False
   (c) Always True

2. (42 points) Evaluate the following expressions:

   (a) $\displaystyle\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\,dx$

$$u = 1+2x \qquad u(0) = 1$$
$$du = 2dx \qquad u(13) = 27$$

$$\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\,dx = \int_1^{27} \frac{1}{2}u^{-2/3}\,du$$
$$= \frac{1}{2}\frac{u^{1/3}}{1/3}\Bigg|_1^{27} = \frac{3}{2}(3-1) = 3$$

   (b) $\displaystyle\int \cos^5\frac{\theta}{2}\,d\theta = \int \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)^2\,d\theta$

$$u = \sin\frac{\theta}{2}$$
$$du = \frac{1}{2}\cos\frac{\theta}{2}d\theta$$

$$\int \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)^2\,d\theta = \int 2\left(1-u^2\right)^2\,du$$
$$= \int 2\left(1-2u^2+u^4\right)\,du = 2\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)+C$$
$$= 2\left(\sin\frac{\theta}{2}-\frac{2}{3}\left(\sin\frac{\theta}{2}\right)^3+\frac{1}{5}\left(\sin\frac{\theta}{2}\right)^5\right)+C$$

   (c) $\displaystyle\int \csc^2\sqrt{x}\sqrt{\frac{\tan\sqrt{x}}{x}}\,dx = \int \csc^2\sqrt{x}\,\frac{1}{\sqrt{x}\sqrt{\cot\sqrt{x}}}\,dx$

$$u = \cot\sqrt{x}$$
$$du = -\frac{\csc^2\sqrt{x}}{2\sqrt{x}}dx$$
$$-2du = \frac{\csc^2\sqrt{x}}{\sqrt{x}}dx$$

$$\int \csc^2\sqrt{x}\,\frac{1}{\sqrt{\cot\sqrt{x}}\sqrt{x}}\,dx = \int -\frac{2}{\sqrt{u}}\,dx$$
$$= -2\frac{u^{1/2}}{1/2}+C$$
$$= -4\sqrt{\cot\sqrt{x}}+C$$

(d) $\int_0^a x^3\sqrt{a^2-x^2}\,dx$

$$\begin{aligned} u &= a^2 - x^2 \\ &\Rightarrow x^2 = a^2 - u \\ du &= -2x\,dx \\ u(0) &= a^2,\ u(a) = 0 \end{aligned}$$

$$\begin{aligned} \int_0^a x^3\sqrt{a^2-x^2}\,dx &= \int_{a^2}^0 -\frac{1}{2}\left(a^2-u\right)\sqrt{u}\,du \\ &= \int_{a^2}^0 -\frac{1}{2}\left(a^2u^{1/2}-u^{3/2}\right)\,du \\ &= -\frac{1}{2}\left(a^2\frac{u^{3/2}}{3/2}-\frac{u^{5/2}}{5/2}\right)\Bigg|_{a^2}^0 \\ &= 0-\left(\frac{1}{5}(a^2)^{5/2}-\frac{1}{3}a^2(a^2)^{3/2}\right) = \frac{2}{15}a^5 \end{aligned}$$

(e) $\int \frac{(2t+1)^2}{\sqrt{t+3}}\,dt$

$$\begin{aligned} u &= t+3 \\ &\Rightarrow 2t+1 = 2u-5 \\ du &= dt \end{aligned}$$

$$\begin{aligned} \int \frac{(2t+1)^2}{\sqrt{t+3}}\,dt &= \int \frac{(2u-5)^2}{\sqrt{u}}du = \int \frac{4u^2-20u+25}{\sqrt{u}}du \\ &= \int \left(4u^{3/2}-20u^{1/2}+25u^{-1/2}\right)du \\ &= 4\frac{u^{5/2}}{5/2}-20\frac{u^{3/2}}{3/2}+25\frac{u^{1/2}}{1/2}+C \\ &= \frac{8}{5}(t+3)^{5/2}-\frac{40}{3}(t+3)^{3/2}+50(t+3)^{1/2}+C \end{aligned}$$

(f) $\sum_{k=1}^{n}(2+(k-1)^3)$

$$= 2n+\sum_{k=2}^{n}(2+(k-1)^3) = 2n+\sum_{k=1}^{n-1}(2+k^3) = 2n+\left(\frac{(n-1)n}{2}\right)^2$$

(g) $\sum_{k=-2}^{3}\sin\frac{k\pi}{3}\cos k\pi$

$$\begin{aligned} &= \sin\left(\frac{-2\pi}{3}\right)\cos(-2\pi)+\sin\left(\frac{-\pi}{3}\right)\cos(-\pi)+\sin 0\cos 0 \\ &\quad +\sin\left(\frac{\pi}{3}\right)\cos\pi+\sin\left(\frac{2\pi}{3}\right)\cos 2\pi+\sin\pi\cos 3\pi \\ &= \left(\frac{-\sqrt{3}}{2}\right)(1)+\left(\frac{-\sqrt{3}}{2}\right)(-1)+(0)(1)+\left(\frac{\sqrt{3}}{2}\right)(-1)+\left(\frac{\sqrt{3}}{2}\right)(1)+(0)(-1) \\ &= 0 \end{aligned}$$

3. (20 points)

(a) The acceleration and initial velocity are given as *downward* information, so we need to take the negative of them to be in the frame of reference where up is positive and down is negative.

$$a(t) = \begin{cases} -9 + 0.9\,t & \text{for} \quad 0 \le t \le 10 \\ 0 & \text{for} \quad t > 10 \end{cases}$$

$$\Rightarrow v(t) = \begin{cases} -9t + 0.45\,t^2 + C_0 & \text{for} \quad 0 \le t \le 10 \\ C_1 & \text{for} \quad t > 10 \end{cases}$$

$$v(0) = -10\,\frac{m}{s} = C_0$$

$$v(t)\text{must be continuous} \Rightarrow v(10) = -90 + 45 - 10 = -55 = C_1$$

$$v(t) = \begin{cases} -9t + 0.45\,t^2 - 10 & \text{for} \quad 0 \le t \le 10 \\ -55 & \text{for} \quad t > 10 \end{cases}$$

$$\Rightarrow s(t) = \begin{cases} -\frac{9}{2}t^2 + 0.15\,t^3 - 10t + C_2 & \text{for} \quad 0 \le t \le 10 \\ -55t + C_3 & \text{for} \quad t > 10 \end{cases}$$

$$s(0) = 500 = C_2$$

$$s(t)\text{must be continuous} \Rightarrow s(10) = -450 + 150 - 100 + 500 = -550 + C_3$$

$$\Rightarrow C_3 = 650$$

$$\Rightarrow s(t) = \begin{cases} -\frac{9}{2}t^2 + 0.15\,t^3 - 10t + 500 & \text{for} \quad 0 \le t \le 10 \\ -55t + 650 & \text{for} \quad t > 10 \end{cases}$$

(b) How long does it take for the raindrop to reach the ground?

$$s(t) = -55t + 650 = 0$$

$$\Rightarrow t = \frac{130}{11}$$

(c) With what velocity does the raindrop strike the ground?

$$v\left(\frac{130}{11}\right) = -55$$

$\Rightarrow$ The raindrop hits the ground with a downwards velocity of 55 m/s.

4. (12 points) If $F(x) = \int_{x-1}^{x+1} f(t)\,dt$ and $f(t) = \int_1^{t^2} \frac{\sqrt{w^2-1}}{w}\,dw$, find $F''(2)$.

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\int_{x-1}^{x+1} f(t)\,dt \;=\; f(x+1) - f(x-1) && \text{(By FTC pt 1)}\\
\Rightarrow F''(x) &= f'(x+1) - f'(x-1)\\
f'(t) &= \frac{d}{dt}\int_1^{t^2} \frac{\sqrt{w^2-1}}{w}\,dw\\
&= \frac{\sqrt{(t^2)^2-1}}{(t^2)}\cdot(2t) \;=\; \frac{2\sqrt{t^4-1}}{t} && \text{(By FTC pt 1)}\\
\Rightarrow F''(x) &= \frac{2\sqrt{(x+1)^4-1}}{x+1} - \frac{2\sqrt{(x-1)^4-1}}{x-1}\\
\Rightarrow F''(2) &= \frac{2\sqrt{3^4-1}}{3} - 0 \;=\; \frac{2\sqrt{80}}{3}
\end{aligned}$$

5. (14 points) Find the Riemann sum for the function $f(x) = 2 + (x-2)^2$ on the interval $[0, 2]$ using 4 subintervals and the left endpoint of each subinterval. Sketch the graph of f and the approximating rectangles.

Intervals: $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right]$

$\Delta x = \frac{1}{2}$

$$\begin{aligned}
\text{A} &= \frac{1}{2}\left(2 + (0-2)^2\right) + \frac{1}{2}\left(2 + \left(\frac{1}{2} - 2\right)^2\right) + \frac{1}{2}\left(2 + (1-2)^2\right) + \frac{1}{2}\left(2 + \left(\frac{3}{2} - 2\right)^2\right)\\
\Rightarrow \text{A} &= 3 + \frac{17}{8} + \frac{3}{2} + \frac{9}{8} = \frac{31}{4}
\end{aligned}$$