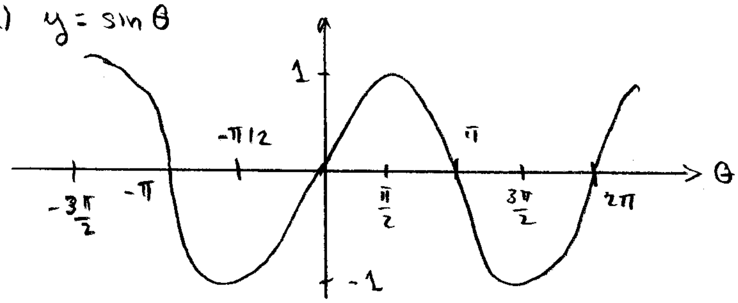
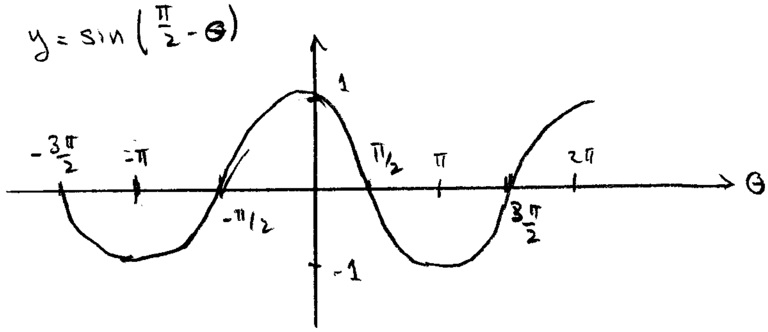
APPM 1350: Forgiveness Exam 1: Solutions

1. Graph the following functions of θ, each on a different coordinate system.

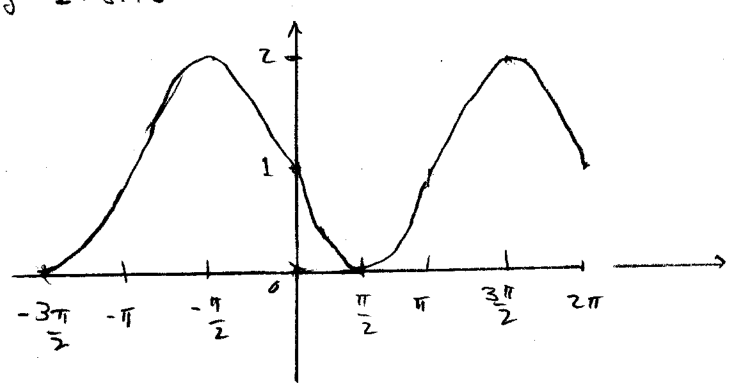
a) $y = \sin\theta$



b) $y = \sin\left(\frac{\pi}{2} - \theta\right)$



c) $y = 1 - \sin\theta$



2. Consider the function $f(x) = \frac{x}{x-1}$ and $g(x) = x+1$

a) What are the domains of f and g?

Soln:

$$\boxed{\begin{array}{l}\text{Domain of } f(x) = \{x : x \in \mathbb{R},\ x \neq 1\} \\ \text{Domain of } g(x) = \mathbb{R}\end{array}}$$

b) Calculate $(g \circ f)(x)$. What is the domain of $(g \circ f)$?

Soln.

$$\boxed{\begin{array}{l}(g \circ f)(x) = \left(\frac{x}{x-1}\right) + 1 \\ \\ \text{Domain} = \{x : x \in \mathbb{R},\ x \neq 1\}\end{array}}$$

c) Calculate $(f \circ f)(x)$. What is the domain of $(f \circ f)$?

Soln:

$$(f \circ f)(x) = \frac{\left(\frac{x}{x-1}\right)}{\left(\frac{x}{x-1} - 1\right)}$$

$$= \left(\frac{x}{x-1}\right)\left(\frac{x-1}{x-x+1}\right)$$

$$= x$$

So $\boxed{\begin{array}{l}(f \circ f)(x) = x \\ \text{Domain} = \mathbb{R}\end{array}}$

3. Consider the function $f(t) = 2\sin\left(\frac{t}{2}\right)$, defined on the interval $[0, 4\pi]$

a) What is the average rate of change of f over the interval $[0, \frac{\pi}{3}]$?

Soln:

$$\text{Avg rate of change} = \frac{f\left(\frac{\pi}{3}\right) - f(0)}{\frac{\pi}{3} - 0}$$

$$= \frac{2\sin\left(\frac{\pi}{6}\right) - 2\sin 0}{\frac{\pi}{3}}$$

$$= \frac{6}{\pi}\sin\left(\frac{\pi}{6}\right)$$

$$= \boxed{\frac{3}{\pi}}$$

b) What is the instantaneous rate of change of f at $t = 0$?

Soln:

$$\text{Instantaneous rate of change} = \left.\frac{d}{dt} f(t)\right|_{t=0}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right) - 2\sin 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$\boxed{= 1}$$

c) What is the equation of the tangent line at the origin to the graph of f?

Soln:

$$\boxed{\begin{aligned} L(x) &= f(0) + f'(0)\,x \\ &= x \end{aligned}}$$

4. Calculate the following limits. If they do not exist, write DNE.

a) $\lim_{x \to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \boxed{\sqrt{2}}$

b) $\lim_{x \to 0^-} \frac{4}{(x^2/5)} = \boxed{\infty}$

c) $\lim_{x \to \sqrt{2}} \left(\frac{x^2}{2} - \frac{1}{x}\right) = \boxed{1 - \frac{1}{\sqrt{2}}}$

d) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x^2(x-2)}$

$= \lim_{x \to 2} \frac{x-1}{x^2}$

$= \boxed{\frac{1}{4}}$

5. a) State clearly the definition of a continuous function

Soln:

$$\boxed{f(x) \text{ is continuous at } x=a \text{ if } f(a) \text{ is defined, } \lim_{x\to a} f(x) \text{ exists, and } \lim_{x\to a} f(x) = f(a)}$$

b) For what value of a is the function

$$f(x) = \begin{cases} x^2-1 & x<3 \\ 2ax & x \geq 3 \end{cases}$$

continuous at every point?

Soln:

$$\lim_{x\to 3^-} x^2-1 = \lim_{x\to 3^+} 2ax$$

$$\Rightarrow 9-1 = 6a$$

$$\Rightarrow \boxed{a = \frac{4}{3}}$$

c) Is there a value of a for which f is differentiable at every point?

Soln:

A differentiable function must be continuous. So the only possible choice for a is 4/3. However, with this choice:

$$f(x) = \begin{cases} x^2-1 & x<3 \\ \frac{8}{3}x & x \geq 3 \end{cases}$$

and $\lim_{x\to 3^-} f'(x) = 6$ while $\lim_{x\to 3^+} f'(x) = \frac{8}{3}$. These are not the same which means $f'(x)$ does not exist. Short answer. $\boxed{\text{No}}$

6. a) For $y = \dfrac{2x+1}{x^2+1}$, calculate $\dfrac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = \frac{(x^2+1)(2) - (2x+1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2+2-4x^2-2x}{(x^2+1)^2}$$

$$= \boxed{\frac{-2x^2-2x+2}{(x^2+1)^2}}$$

b) For $y = x^3 + \dfrac{1}{x+1}$, calculate $\dfrac{d^2y}{dx^2}$

Solution:

$$\frac{dy}{dx} = 3x^2 - \frac{1}{(x+1)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = 6x + \frac{2}{(x+1)^3}}$$

7. a) Using the definition, calculate the derivative of the function $y = \sqrt{x+1}$ for $x > -1$.

Solution:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h+1} - \sqrt{x+1}\right)\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \lim_{h \to 0} \frac{(\cancel{x}+\cancel{h}+\cancel{1}) - (\cancel{x}+\cancel{1})}{\cancel{h}\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \boxed{\frac{1}{2\sqrt{x+1}}}$$

b) At what point(s) on the curve in (a) does the tangent line have slope $m = 2$?

Solution:

$$\frac{1}{2\sqrt{x+1}} = 2$$

$$\Rightarrow \sqrt{x+1} = \frac{1}{4}$$

$$\Rightarrow x = -1 + \frac{1}{16}$$

$$= \boxed{-\frac{15}{16}}$$

Joker problem: We suspect that 3 particles originated as by-products of the same molecular explosion, but we are not certain of what happened at the time of the explosion. The particles move along the same line. Particle X is oscillating, so that at time t after its birth it is $f(t) = \frac{\sin t}{t}$ microns away from a reference point 0. Particle Z is slowly moving away, so that at time t it is $g(t) = t+1$ from 0. All we know about particle Y is that it is always trapped between X and Z. Is our suspicion of their common origin correct? (Hint: You want to check that the past trajectories of all particles approach the same point as $t \to 0$).

Solution:

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{\sin t}{t} = 1$$

$$\lim_{t \to 0} g(t) = \lim_{t \to 0} (t+1) = 1$$

Let $h(t)$ be the trajectory of Y. Since $f(t) \leq h(t) \leq g(t)$, from the sandwich theorem:

$$\lim_{t \to 0} f(t) \leq \lim_{t \to 0} h(t) \leq \lim_{t \to 0} g(t)$$

$$\Rightarrow \quad 1 \leq \lim_{t \to 0} h(t) \leq 1$$

$$\Rightarrow \quad \lim_{t \to 0} h(t) = 1$$

All the limits at $t=0$ are the same, so $\boxed{\text{YES}}$, all particles originated from the same location.