APPM 1350: Forgiveness Exam 2: Solutions

1.a) Is there a value of b that would make the function

$$f(x) = \begin{cases} x+b & x<0 \\ \cos(x) & x \geq 0 \end{cases}$$

differentiable at $x=0$?

Soln: Continuity requires $b = \cos 0 = 1$.

$$f'(x) = \begin{cases} 1 & x<0 \\ 0 & x \geq 0 \end{cases} \Rightarrow f'(x) \text{ DNE at } x=0$$

$\boxed{\text{No, there is no } b \text{ such that } f(x) \text{ is differentiable at } x=0}$

b) Calculate $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$

Soln: $$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{x} \lim_{x\to 0} \frac{1}{\cos 2x}$$

$$= 2 \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right) \lim_{x \to 0} \sec 2x$$

$$= \boxed{2}$$

c) Calculate $\lim\limits_{x \to -\infty} \dfrac{x^4 + 3x + 1}{x^2 + 1}$

Soln:

$$\lim_{x \to -\infty} \frac{x^4 + 3x + 1}{x^2 + 1} = \lim_{x \to -\infty} \frac{x^4\left(1 + \frac{3}{x^3} + \frac{1}{x^4}\right)}{x^2\left(1 + \frac{1}{x^2}\right)}$$

$$= \lim_{x \to -\infty} x^2 \lim_{x \to -\infty} \frac{1 + \frac{3}{x^3} + \frac{1}{x^4}}{1 + \frac{1}{x^2}}$$

$$= \infty \cdot 1$$

$$= \boxed{\infty}$$

d) Use implicit differentiatren to find $\frac{dy}{dx}$ given $x^2 = \dfrac{x-y}{x+y}$

Soln:

$$2x = \frac{(x+y)\left(1 - \frac{dy}{dx}\right) - (x-y)\left(1 + \frac{dy}{dx}\right)}{(x+y)^2}$$

$$\Rightarrow 2x(x+y)^2 = \cancel{x} + y - \frac{dy}{dx}(x+y) - \cancel{x} + y - (x-y)\frac{dy}{dx}$$

$$\Rightarrow 2x(x+y)^2 = 2y - \frac{dy}{dx}(x + \cancel{y} + x - \cancel{y})$$

$$\Rightarrow 2x(x+y)^2 = 2y - 2x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\left[y - x(x+y)^2\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x} - (x+y)^2}$$

(see next page)

Alternately, start with

$$x^2 = \frac{x-y}{x+y}$$

$$\Rightarrow x^2(x+y) = x-y$$

$$\Rightarrow x^3 + x^2 y = x - y$$

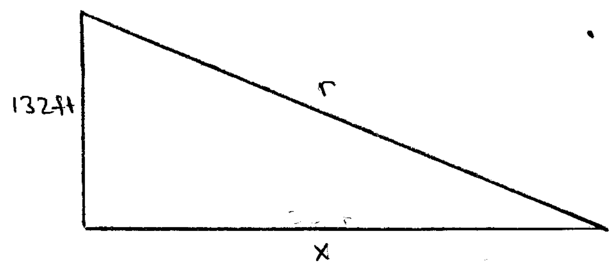
$$\Rightarrow 3x^2 + 2xy + x^2 \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow 3x^2 + 2xy + \frac{dy}{dx}(x^2+1) = 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1 - 3x^2 - 2xy}{x^2+1}}$$

2. A cop is trying to videotape a robber who has stolen a Porche. Unknown to her, there is an automatic camera placed on a pole 132 feet above the ground tracking her movements. The car is moving directly towards the pole at 264 ft/sec. What would the speed of the car with respect to the camera be when the car is 30 ft from the base of the pole?

Solution:



We are given $\frac{dx}{dt} = -264$ ft/sec and we want $\frac{dr}{dt}$ when $x = 30$.

So:

$$r^2 = x^2 + (132)^2$$

$$\Rightarrow 2r\frac{dr}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{x}{r}\frac{dx}{dt}$$

$$= \frac{x}{\sqrt{x^2 + (132)^2}}\frac{dx}{dt}$$

When $x = 30$ and $\frac{dx}{dt} = -264$:

$$\boxed{\frac{dr}{dt} = \frac{30}{\sqrt{(30)^2 + (132)^2}}(-264) \approx -58.5 \text{ ft/sec}}$$

3. Consider the equation $x = \dfrac{1}{x^2+1}$

a) Show that the equation has at least one solution in $(0,1)$

Soln: Let $y = x - \frac{1}{x^2+1}$. Then $y(0) = -1$ and $y(1) = \frac{1}{2}$. Since $y(x)$ is continuous, then $\exists\, c \in (0,1) \ni y(c) = 0$ by the intermediate value theorem.

b) Approximate the solution by using Newton's method for $x_0 = 1$ (calculate only x_1)

Soln:

$$x_n = x_{n-1} - \frac{y(x_{n-1})}{y'(x_{n-1})}$$

$$y(x_0) = y(1) = \frac{1}{2}$$

$$y'(x_0) = 1 + \frac{2x_0}{(x_0^2+1)^2}$$

$$= 1 + \frac{2}{4}$$

$$= \frac{3}{2}$$

$$\Rightarrow x_1 = 1 - \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)}$$

$$= 1 - \frac{1}{3}$$

$$= \boxed{\frac{2}{3}}$$

4. Consider the function $f(x) = \dfrac{x^2}{x-1}$ for $x \neq 1$.

a) Find the x and y coordinates of all local maxima and minima. Identify which are maxima and which are minima. Justify your answers.

Soln: $$f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2}$$

The critical points are when $f'(x) = 0 \Rightarrow x = 0$ and $x = 2$.

Note:

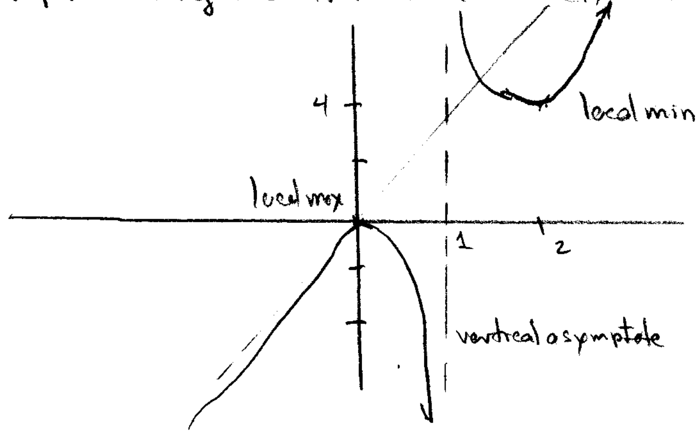
$$\left.\begin{array}{lll} x < 0 & \Rightarrow & f'(x) > 0 \\ 0 < x < 1 & \Rightarrow & f'(x) < 0 \end{array}\right\} \Rightarrow \boxed{x = 0 \text{ in a local max} \Rightarrow (0,0)}$$

$$\left.\begin{array}{lll} 1 < x < 2 & \Rightarrow & f'(x) < 0 \\ x > 2 & \Rightarrow & f'(x) > 0 \end{array}\right\} \Rightarrow \boxed{x = 2 \text{ is a local min} \Rightarrow (2,4)}$$

b) Determine any horizontal or vertical asymptotes the graph of f might have.

Soln: Vertical asymptote at $x = 1$, $f(x) \to \pm\infty$ as $x \to 1$
No horizontal asymptotes

c) Graph f using the information from (a) and (b)



$f(0) = 0$

$f(2) = 4$

5. a) State the mean value theorem

Soln:

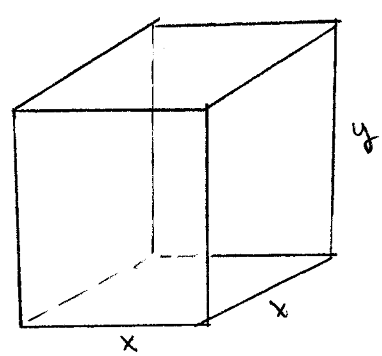
If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there is a number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

b) The US postal service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimension will give a box with a square end and the largest possible volume?

Soln:

lenth = y
girth = $4x$



Hence $y+4x = 108$ and $V = x^2 y$. Then

$$\begin{aligned} V &= x^2 y \\ &= x^2(108-4x) \\ &= 108x^2 - 4x^3 \end{aligned}$$

$$\frac{dV}{dx} = 216x - 12x^2 = 0$$

$$\Rightarrow x(216-12x) = 0$$

$$\Rightarrow 12x(18-x) = 0$$

$$\Rightarrow \boxed{x = 18\text{ in} \qquad y = 36\text{ in}}$$