Final Exam

① a) $$\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{2x}}{(x-1)} \left(\frac{\sqrt{x^2+1}+\sqrt{2x}}{\sqrt{x^2+1}+\sqrt{2x}}\right)$$ +3

$$\lim_{x\to 1} \frac{x^2+1-2x}{(x-1)(\sqrt{x^2+1}+\sqrt{2x})}$$

$$\lim_{x\to 1} \frac{x^2-2x+1}{(x-1)(\sqrt{x^2+1}+\sqrt{2x})}$$ +3

$$\lim_{x\to 1} \frac{(x-1)^2}{(x-1)(\sqrt{x^2+1}+\sqrt{2x})}$$ +3

$$\lim_{x\to 1} \left(\frac{x-1}{\sqrt{x^2+1}+\sqrt{2x}}\right) = \frac{0}{\sqrt{2}+\sqrt{2}} = \boxed{0}$$ +3 +3

or

L'Hopital

$$\lim_{x\to 1} \underbrace{\frac{2x}{2\sqrt{x^2+1}}}_{+5} - \underbrace{\frac{2}{2\sqrt{2x}}}_{+5} = \frac{2}{2\sqrt{2}} - \frac{2}{2\sqrt{2}} = \boxed{0}$$ +5

b) $$\lim_{x\to 0} \frac{x-\sin x}{1-\cos x} = \frac{0-0}{1-1} = \frac{0}{0} \text{ und}$$

Use L'Hopitals

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \frac{1-1}{0} = \frac{0}{0} \text{ und}$$ +2 +2 +2 +2

Do again

$$\lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = \boxed{0}$$ +2 +2 +3

c) $$\lim_{x\to\infty} \frac{x^6}{6e^x} = \frac{\infty}{\infty} \text{ und}$$

Use L'Hopitals

$$\lim_{x\to\infty} \frac{6x^5}{6e^x} = \lim_{x\to\infty} \frac{x^5}{e^x} = \frac{\infty}{\infty}$$ +2

Do again 5 more times

$$\lim_{x\to\infty} \underbrace{\frac{5x^4}{e^x}}_{+2} = \lim_{x\to\infty} \underbrace{\frac{20x^3}{e^x}}_{+2} = \lim_{x\to\infty} \underbrace{\frac{60x^2}{e^x}}_{+2} = \lim_{x\to\infty} \underbrace{\frac{120x}{e^x}}_{+2} = \lim_{x\to\infty} \underbrace{\frac{120}{e^x}}_{+2} = \boxed{0}$$ +3

② a) $y = (1+2x)e^{-2x}$

$y' = \overset{+2}{2}e^{\overset{+2}{-2x}} + \overset{+2}{(1+2x)}\overset{+2}{(-2e^{-2x})}$

+2 $y' = 2e^{-2x} - 2e^{-2x} - 4xe^{-2x}$

+5 $\boxed{y' = -4xe^{-2x}}$

or

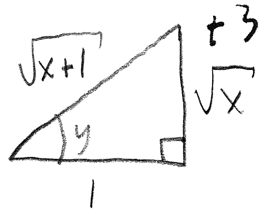
$y = \overset{+2}{e^{-2x}} + \overset{+2}{2xe^{-2x}}$

$y' = \overset{+2}{-2e^{-2x}} + \overset{+2}{2e^{-2x}} - \overset{+2}{2(2x)e^{-2x}}$

$\boxed{y' = -4xe^{-2x}}$ +5

b) $y = \tan^{-1}(\sqrt{x})$

+3 $\tan y = \sqrt{x}$



Implicit Differentiation

+3 $\sec^2 y \, y' = \frac{1}{2\sqrt{x}}$

+3 $y' = \frac{1}{2\sec^2 y(\sqrt{x})}$

+3 $\boxed{y' = \frac{1}{2(x+1)\sqrt{x}}}$

c) $y = x^x$

+4 $\ln y = x \ln x$

Implicit Diff.

$\overset{+2}{\frac{1}{y}y'} = \overset{+2}{\ln x} + \overset{+2}{x\left(\frac{1}{x}\right)}$

$\frac{1}{y}y' = \ln x + 1$ +1

$y' = y(\ln x + 1)$ +1

+3 $\boxed{y' = x^x(\ln x + 1)}$

3

a) $\int e^{(x^2-4x+7)}(x-2)\,dx$

+4 $u = x^2-4x+7$

+2 $du = (2x-4)\,dx$

+2 $du = 2(x-2)\,dx$

+3

$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u = \boxed{\frac{1}{2}e^{(x^2-4x+7)} + C}$

+2 +2

b) $\int_2^6 \frac{4x+4}{x^2+2x-3}\,dx$

+2 $u = x^2+2x-3$

+2 $du = (2x+2)\,dx \to 2du = (4x+4)\,dx$

+2 +2

$\int \frac{2du}{u} = 2\ln|u| = 2\ln|x^2+2x-3|\Big|_2^6 = 2\ln|45| - 2\ln|5|$

+2 +2

$= 2\ln|9| + 2\ln|5| - 2\ln|5|$

$= 2\ln 9$ +3

$= \boxed{\ln 81}$

c) $f(x) = \sin(2x)$

+7 shaded area $= 4\int_0^{\pi/2} \sin(2x)\,dx$

$= -2\cos(2x)\Big|_0^{\pi/2} = -2\cos\pi + 2\cos(0) = -2(-1) + 2(1) = \boxed{4}$

+2 +2 +2 +2

④ $f(x) = \begin{cases} x^2+ax+b & x<0 \\ \sin x & x \geq 0 \end{cases}$

Continuity

Limit from Left = Limit from Right +5

$x^2+ax+b = \sin x$ at $x=0$

$0^2+a(0)+b = \sin 0$ +5

$\boxed{b=0}$

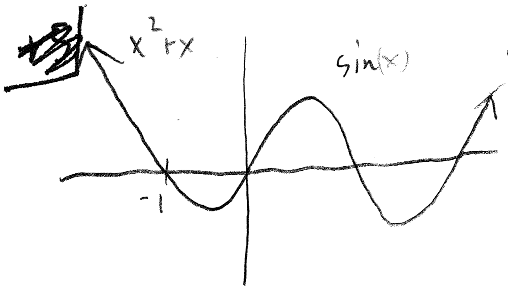
Differentiable

Derivative from Left = Derivative from right +5

$\overset{+2}{2x} + \overset{+2}{a} = \overset{+2}{\cos x}$ at $x=0$

$2(0)+a = \cos(0)$

$\boxed{a=1}$ +5



+4

$f(x) = x^2+x = x(x+1)$

5 $A = ab$

want to maximize

$3a + b = 60$ - equation for perimeter

+4 $\boxed{b = 60 - 3a}$

$A = a(60-3a)$

+4 $A = 60a - 3a^2$

$dA/da = \overset{+2}{60} - \overset{+2}{6a} = 0$

$\frac{6a}{6} = \frac{60}{6}$

+4 $\boxed{a = 10}$

$b = 60 - 3a = 60 - 3(10) = 30$

+4 $\boxed{b = 30}$

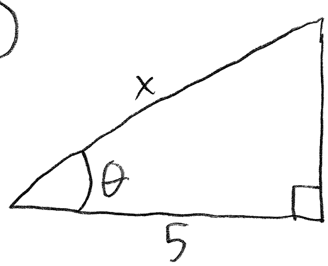
+5 $\boxed{\text{Max Area} = 10(30) = 300 \text{ ft}^2}$

Note:

$\frac{d^2A}{da^2} = -6$ which is concave down

+5 Thus what I found is truly a maximum

6

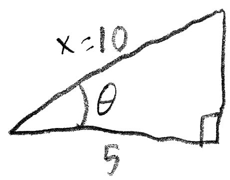


+4 $\cos\theta = \dfrac{5}{x}$

Implicit Diff.

$$\underset{+2}{-\sin\theta}\ \underset{+2}{\frac{d\theta}{dt}} = \overset{+2}{\frac{-5}{x^2}}\ \overset{+2}{\frac{dx}{dt}}$$

At the given moment $x = 10$



$$\cos\theta = \frac{5}{10}$$

Thus $\theta = \pi/3$ +4

$$-\sin(\pi/3)\left(\frac{20}{\sqrt{3}}\right) = \frac{-5}{10^2}\ dx/dt$$

$$-\overset{+2}{\left(\frac{\sqrt{3}}{2}\right)}\overset{+1}{\left(\frac{20}{\sqrt{3}}\right)} = \overset{+1}{\frac{-5}{100}}\ dx/dt$$

$$\overset{+2}{-10} = \overset{+2}{-\frac{1}{20}}\ dx/dt$$

+6 $\boxed{dx/dt = 200 \text{ ft/min}}$

+4 +2