APPM 1350 Final Exam – Answers

1 a) $f(x)$ is continuous at $x = x_0$ if:

- $f(x_0)$ is defined
- $\lim_{x \to x_0} f(x)$ exists (so $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$)
- $\lim_{x \to x_0} f(x) = f(x_0)$

b) $g(x) = x\cos(\frac{1}{x})$, $x \neq 0$

$= 1$, $x = 0$

sandwich Th'm: $-1 \le \cos(\frac{1}{x}) \le 1$

$\Rightarrow -|x| \le x\cos(\frac{1}{x}) \le |x|$

$\Rightarrow \lim_{x\to 0}[-|x|] \le \lim_{x\to 0} x\cos(\frac{1}{x}) \le \lim_{x \to 0} |x|$

$\Rightarrow \lim_{x\to 0} x\cos(\frac{1}{x}) = 0 \neq g(0) = 1 \Rightarrow$ <u>NOT</u> continuous

c) $f(x) = \frac{1}{x-2}$ is continuous except at $x=2$

$g(x) = x+2$ is continuous for all x

$f \circ g(x) = f(g(x)) = \frac{1}{(x+2)-2} = \frac{1}{x}$ is continuous except at $x=0$

d) See (c)

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2. a) $F(x) = \ln(1+x)$, $F'(x) = \frac{1}{1+x}$, $F''(x) = -\frac{1}{(1+x)^2}$

$F'''(x) = \frac{2}{(1+x)^3}$

at $x=0$, $F(0)=0$, $F'(0)=1$

$L(x) = 0 + (1)(x-0) \Rightarrow \boxed{L(x) = x}$

b) For $\ln(1+x) = \ln(1.1) \Rightarrow x = 0.1 \Rightarrow \underline{L(0.1) = 0.1}$

c) $\ln(1+x) = \int_1^{1+x} \frac{1}{t}\,dt$

d) Trapezoidal rule: $T = \frac{0.1}{2}\left[\frac{1}{1} + \frac{1}{1.1}\right] = \underline{(0.1)(\frac{21}{22})} = 0.09545\ldots$

e) Error: $|E_T| \le \frac{(0.1)^3(1)}{12}(2) = \underline{\frac{1}{6000}}$ Exact: $\ln(1.1) = 0.0953102\ldots$

3. a) $z = \ln(\tan^{-1}(x)) \Rightarrow \dfrac{dz}{dx} = \dfrac{1}{\tan^{-1}(x)} \cdot \dfrac{1}{1+x^2}$

b) $y = \tan^{-1}(\ln(x)) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+(\ln(x))^2} \cdot \dfrac{1}{x}$

c) $z(1) = \ln(\tan^{-1}(1)) = \ln(\frac{\pi}{4}) < 0$ because $0 < \frac{\pi}{4} < 1$

$y(1) = \tan^{-1}(\ln(1)) = \tan^{-1}(0) = 0$

$\Rightarrow \underline{z(1) < y(1)}$

d) Tangent Line at $x=1$: $Y(x) = y(1) + \frac{dy}{dx}(1) \cdot (x-1)$

$y(1) = 0, \quad \frac{dy}{dx}(1) = \frac{1}{1+0} \cdot \frac{1}{1} = 1 \Rightarrow \underline{Y(x) = x-1}$

e) $I = \displaystyle\int_{-2}^{2} \frac{dt}{4+3t^2} = \frac{1}{3}\int_{-2}^{2} \frac{dt}{(\frac{4}{3}) + t^2} \Rightarrow a^2 = \frac{4}{3}$ in formula

$I = \frac{1}{3} \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{t}{(2/\sqrt{3})}\right)\Big|_{-2}^{2} = \frac{1}{2\sqrt{3}}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(-\sqrt{3})\right]$

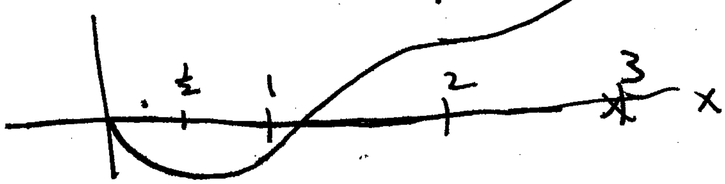
$= \frac{1}{2\sqrt{3}}\left[\frac{\pi}{3} - (-\frac{\pi}{3})\right] = \boxed{\frac{\pi}{3\sqrt{3}}}$

4. a) $g(x)$ has Local maxima at the endpoints ($x=0$, $x=3$)
$g(x)$ has a local minimum at $x = \frac{1}{2}$

b) $g(x)$ is concave up in $0<x<1$, $2<x<3$
concave down in $1<x<2$

c) $g(x)$ has inflection points at $x=1$, $x=2$

d)



5. $A = s^2 + \pi r^2$

$4 = 4s + 2\pi r \Rightarrow s = 1 - \frac{\pi}{2} r$

$\cancel{A = s^2 + \pi t}$ $\quad A = (1 - \frac{\pi}{2} r)^2 + \pi r^2$

$\dfrac{dA}{dr} = 2(1 - \frac{\pi}{2}r)(-\frac{\pi}{2}) + 2\pi r = 0 \Rightarrow \underbrace{r = \dfrac{1}{2(1+\frac{\pi}{4})}, \; s = \dfrac{4}{4+\pi}}_{\text{only critical point.}}$

$\dfrac{d^2A}{dr^2} = \dfrac{\pi^2}{2} + 2\pi > 0 \Rightarrow \underline{\underline{\text{minimum}}}$

$\Rightarrow$ Maximum at end-point $\begin{cases} r = 0 \Rightarrow s = 1 \\ \Rightarrow A_{\square} = 1 \end{cases}$ $\begin{cases} \boxed{s = 0 \Rightarrow r = 2/\pi} \\ \Rightarrow A_{\bigcirc} = \pi(\frac{2}{\pi})^2 = \boxed{\frac{4}{\pi}} \text{ max.} \end{cases}$

6) a) $e^x = 2 \Rightarrow \underline{x = Ln(2)}$; $2^y = e \Rightarrow$
$(e^{Ln 2})^y = e \Rightarrow y\,Ln(2) = 1$
$\Rightarrow \underline{y = \frac{1}{Ln(2)}}$

b) $\underline{xy = 1}$

c) $e^{5x} = (e^x)^5 = 2^5 = \boxed{32}$

d) $I = \int_1^4 \frac{2^{\sqrt{t}}}{\sqrt{t}}\,dt$: set $u = \sqrt{t}$, $du = \frac{1}{2\sqrt{t}}\,dt \Rightarrow \frac{dt}{\sqrt{t}} = 2du$

$$I = 2\int_1^2 2^u\,du = 2\int_1^2 e^{u\,Ln(2)}\,du = \frac{2e^{u\,Ln(2)}}{Ln(2)}\Big|_1^2$$

$$= \frac{2}{Ln(2)}\left[e^{2Ln(2)} - e^{Ln(2)}\right] = \frac{2}{Ln(2)}[4-2] = \boxed{\frac{4}{Ln(2)}}$$

e) $Ln(10\,Ln(z)) = Ln(5t^2) \Rightarrow 10\,Ln(z) = (5t^2) \Rightarrow Ln(z) = \frac{t^2}{2}$
$\Rightarrow \boxed{z(t) = e^{t^2/2}}$

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7) $y(x) = x^{\frac{1}{x-1}}$, $x > 0$

a) $y(2) = 2^1 = 2$, $y(3) = 3^{\frac{1}{2}} = \sqrt{3}$

b) $\lim_{x\to 1^+} y(x)$: Write $y(x) = (e^{Ln(x)})^{\frac{1}{x-1}} = e^{\left(\frac{Ln(x)}{x-1}\right)}$
as $x \to 1^+$, $\frac{Ln(x)}{x-1} \to$ "$\frac{0}{0}$" $\Rightarrow \lim_{x\to 1^+} \frac{Ln(x)}{x-1} = \lim_{x\to 1^+} \frac{\left(\frac{1}{x}\right)}{1} = 1$

$\therefore \lim_{x\to 1^+} e^{\frac{Ln(x)}{x-1}} = e^1 = \boxed{e}$

c) $\lim_{x\to\infty} y(x)$ requires! $\lim_{x\to\infty} \frac{Ln(x)}{x-1} =$ "$\frac{\infty}{\infty}$" $= \lim_{x\to\infty} \frac{1}{x} = 0$

$\Rightarrow \lim_{x\to\infty} y(x) = e^0 = \boxed{1}$

d) $\frac{dy}{dx} = e^{\frac{Ln(x)}{x-1}}\left[\frac{1}{x(x-1)} - \frac{Ln(x)}{(x-1)^2}\right]$

Homework problems on exam:

1(e): p. 97, #56
2 (a, b): p. 466, #83
4: p. 244, #23a
6(d): p. 481, #52
7(b): p. 496, #44
3(b, e): p. 518, #14, 32