1a. No, because a function is differentiable at a point if and only if the left-hand and right-hand derivatives there are equal. For this problem, the left-hand derivative of f is

$$\frac{d}{dx}(x+b) = 1,$$

and the right-hand derivative at $x = 0$ is

$$\frac{d}{dx}(\cos x) = -\sin x = -\sin 0 = 0.$$

Since the two derivatives are not equal, there is no value b that would make $f(x)$ differentiable at $x = 0$.

1b.

$$\begin{aligned}
\lim_{x\to 0} x\cot x &= \lim_{x\to 0} x \cdot \frac{\cos x}{\sin x} \\
&= \lim_{x\to 0} \frac{x}{\sin x} \cdot \cos x \\
&= \lim_{x\to 0} \frac{1}{(\sin x)/x} \cdot \cos x \\
&= \lim_{x\to 0} \frac{1}{1} \cdot 1 = 1.
\end{aligned}$$

1c.

$$\lim_{x\to\infty} \frac{x^3+5x+\sin x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{x^3}{x^2}+\frac{5x}{x^2}+\frac{\sin x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x+0+0}{1+0} = \infty.$$

2.

$$\begin{aligned}
\cos r + \cos\theta &= r\theta \\
-\sin r\frac{dr}{d\theta} - \sin\theta &= r + \theta\frac{dr}{d\theta} \\
\frac{dr}{d\theta}(\theta + \sin r) &= -r - \sin\theta \\
\frac{dr}{d\theta} &= \frac{-(r+\sin\theta)}{\theta+\sin r}.
\end{aligned}$$

$$\frac{d^2r}{d\theta^2} = \frac{(\theta+\sin r)(-dr/d\theta - \cos\theta) + (r+\sin\theta)(1+\cos r)}{(\theta+\sin r)^2}.$$

3. Let r represent the total radius of the sphere, V represent the volume of the ice, and S represent the surface area of the ice. We are given that $dV/dt = -10\,\mathrm{in}^3/\mathrm{min}$. To find the rate at which the ice is decreasing when it is 2 in thick, we solve for dr/dt when $r = 6$ in. The formula for the volume of a sphere is $\frac{4\pi}{3}R^3$ so the volume of the ice is

$$\begin{aligned} V &= \frac{4\pi}{3}r^3 - \frac{4\pi}{3}(4^3), \\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt}, \\ -10 &= 4\pi(6)^2\frac{dr}{dt}, \\ \frac{dr}{dt} &= -\frac{10}{4\pi(36)} \\ &= -\frac{5}{72\pi}\ \mathrm{in/min}. \end{aligned}$$

We differentiate the formula $S = 4\pi R^2$ with respect to time t to find the rate at which the surface area is decreasing when $r = 6$ in.

$$\begin{aligned} S &= 4\pi r^2, \\ \frac{dS}{dt} &= 8\pi r\frac{dr}{dt} = 8\pi(6)\left(-\frac{5}{72\pi}\right) \\ &= -\frac{10}{3}\ \mathrm{in}^2/\mathrm{min}. \end{aligned}$$

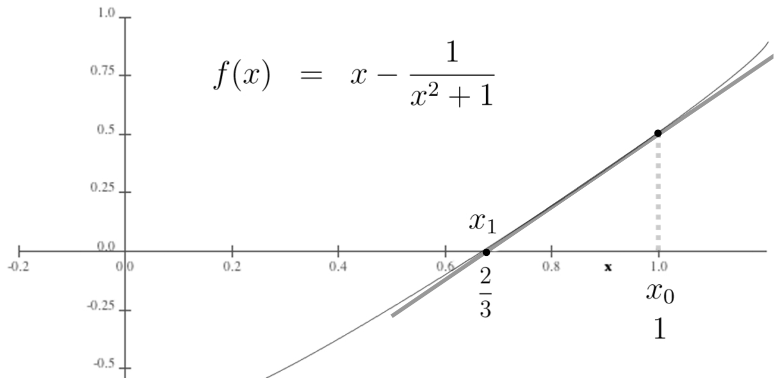
4.

$$\begin{aligned} f(x) &= x - \frac{1}{x^2+1} \\ &= x - (x^2+1)^{-1} \\ f'(x) &= 1 + \frac{2x}{(x^2+1)^2}. \end{aligned}$$

a. We observe that $f(0) = -1$ is negative and $f(1) = 1/2$ is positive and $f(x)$ is continuous. By the Intermediate Value Theorem, $f(x)$ must cross the x-axis in the interval $(0, 1)$. In other words, there must exist a root c in $(0, 1)$ such that $f(c) = 0$.

b. We calculate x_1 given $x_0 = 1$:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1/2}{3/2} = \frac{2}{3}. \end{aligned}$$

c. 

5. For $x \neq \pm 2$,

$$\begin{aligned} f(x) &= \frac{x^2+4x+4}{x^2-4} = \frac{(x+2)(x+2)}{(x+2)(x-2)} = \frac{x+2}{x-2} \\ &= 1 + \frac{4}{x-2} \\ &= 1 + 4(x-2)^{-1}, \end{aligned}$$

$$f'(x) = -4(x-2)^{-2} = -\frac{4}{(x-2)^2}.$$

a. There is a “hole” at $x = -2, y = 0$. We examine the critical points where $f'(x) = 0$ or is undefined. $x = 2$ is a critical point because $f'(x)$ is undefined there. $f'(x)$ never equals 0.

The two intervals we need to examine are $(-\infty, 2)$ and $(2, \infty)$. Since $f'(x)$ is negative for all x, $f(x)$ is decreasing in both intervals. Near the critical point $x = 2$, f approaches $\pm\infty$:

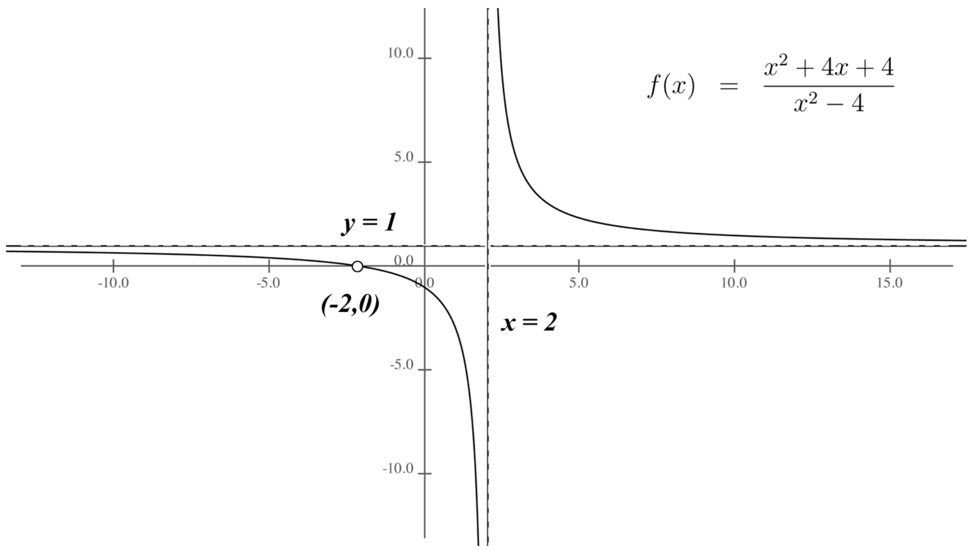
$$\lim_{x \to 2^-} \frac{x+2}{x-2} = -\infty \qquad \lim_{x \to 2^+} \frac{x+2}{x-2} = +\infty$$

Therefore there are no local or absolute extrema.

b. From the limits in 5a, we know that $x = 2$ is a vertical asymptote. For horizontal asymptotes, we check the limits as x approaches $\pm\infty$:

$$\lim_{x \to \infty} 1 + \frac{4}{x-2} = 1 \qquad \lim_{x \to -\infty} 1 + \frac{4}{x-2} = 1$$

Therefore $y = 1$ is a horizontal asymptote.



6a. If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

6b. Let w represent the width and l represent the length of each pen. If the adjacent pens are lined up along the short ends, then the total perimeter $P = 5l + 6w = 240$ ft. The area of each pen is $lw\,\text{ft}^2$ so the total area of the five pens is $A = 5lw\,\text{ft}^2$. We wish to maximize A.

$$\begin{aligned} P &= 5l + 6w = 240 \\ 5l &= 240 - 6w \end{aligned}$$

$$\begin{aligned} A &= 5lw = (240 - 6w)w = 240w - 6w^2, \\ A' &= 240 - 12w \end{aligned}$$

We examine the critical point where $A' = 0$, which occurs when $w = 20$ ft. The endpoints, $w = 0$ and $w = 40$, both result in an area of 0. Since $A'' = -12$, the area function is concave down. Therefore $w = 20$ is a maximum point, and the dimensions $w = 20$ ft, $l = 24$ ft will yield the maximum area of $2400\,\text{ft}^2$.