APPM 1350 – Midterm III solutions

(1)(a) $T_n = \frac{h}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n)$

$T_4 = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$

Here, $h = \frac{1-0}{4} = \frac{1}{4}$, so the partition is: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

$T_4 = \frac{1}{8}\left(\sin(0\pi) + 2\sin\left(\frac{1}{4}\pi\right) + 2\sin\left(\frac{1}{2}\pi\right) + 2\sin\left(\frac{3}{4}\pi\right) + \sin(1\cdot\pi)\right)$

$= \frac{1}{8}\left(0 + \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}} + 2\cdot 1 + \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}} + 0\right) = \frac{1}{8}(2 + 2\sqrt{2}) = \boxed{\frac{1+\sqrt{2}}{4}}$

(b) To be sure that the error is smaller than 0.01, we ask that:

$$\frac{b-a}{12}\cdot h^2 \cdot M \le 0.01$$

Here: $a=0$, $b=1$ and $M = \pi^2$, because:

$$f(x) = \sin(\pi x) \Rightarrow f'(x) = \pi\cos(\pi x) \Rightarrow f''(x) = -\pi^2\sin(\pi x)$$

$$\Rightarrow |f''(x)| = |-\pi^2 \sin \pi x|$$

$$M = \max_{[0,1]} |f''(x)| = \max_{[0,1]} \pi^2 |\sin \pi x| = \pi^2 \cdot 1 = \pi^2$$

So we want that: $\frac{1-0}{12}\cdot h^2 \cdot \pi^2 \le 0.01 \Rightarrow$

$$h^2 \le \frac{12 \times 0.01}{\pi^2}$$

$$\frac{1}{n} = h \le \sqrt{\frac{12\cdot 0.01}{\pi^2}}$$

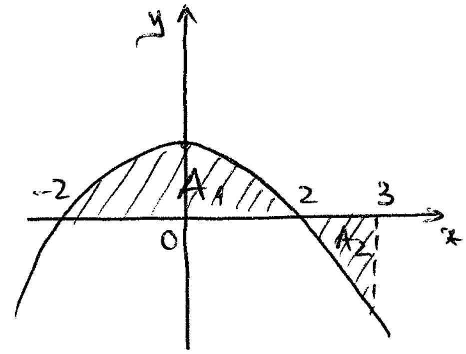
$$\Rightarrow \quad n \ge \sqrt{\frac{\pi^2}{12\cdot 0.01}} = \frac{10\pi}{\sqrt{12}} = \boxed{\frac{5\pi}{\sqrt{3}}}$$

(2) We want to see where $y = 1 - \frac{x^2}{4}$ is positive and where it is negative.

$$1 - \frac{x^2}{4} = 0 \iff \frac{x^2}{4} = 1 \iff x^2 = 4 \iff x = \pm 2$$

$1 - \frac{x^2}{4} \geq 0$ for x in $[-2, 2]$

$1 - \frac{x^2}{4} \leq 0$ for x in $[2, 3]$



$$A_1 = \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right) dx =$$

$$= \left(x - \frac{x^3}{12}\right)\Big|_{-2}^{2} = \left(2 - \frac{8}{12}\right) - \left(-2 + \frac{8}{12}\right) = 4 - \frac{16}{12} = \cancel{2\frac{8}{12}} = \frac{16}{12} = \frac{8}{3}$$

$$A_2 = -\int_{2}^{3} \left(1 - \frac{x^2}{4}\right) dx = -\left(x - \frac{x^3}{12}\right)\Big|_{2}^{3} = -\left[\left(3 - \frac{27}{12}\right) - \left(2 - \frac{8}{12}\right)\right] =$$

$$= -\left(1 - \frac{15}{12}\right) = -\left(-\frac{7}{12}\right) = \frac{7}{12}$$

Conclusion: the area is $A_1 + A_2 = \frac{8}{3} + \frac{7}{12} = \frac{32+7}{12} = \frac{39}{12} = \boxed{\frac{13}{4}}$

(3) (a) $$\int \frac{(1+\sqrt{t})^{1/2}}{\sqrt{t}}\, dt = \int \frac{u^{1/2}}{\cancel{\sqrt{t}}} \cdot 2\cancel{\sqrt{t}}\, du = 2 \cdot \frac{u^{3/2}}{3/2} + C =$$

$u = 1 + \sqrt{t}$

$du = \frac{1}{2\sqrt{t}}\, dt$

$dt = 2\sqrt{t}\, du$

$$= \boxed{\frac{4}{3}(1+\sqrt{t})^{3/2} + C}$$

(b) $\int \frac{\pi}{2} \cos x \sin(\pi + \pi \sin x)\, dx = \int \frac{\pi}{2} \cancel{\cos x} \sin u \cdot \frac{du}{\pi \cancel{\cos x}} =$

$u = \pi + \pi \sin x$

$du = \pi \cos x \, dx$

$dx = \frac{du}{\pi \cos x}$

$= \frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C$

$= \boxed{-\frac{1}{2} \cos(\pi + \pi \sin x) + C}$

(c) $\int_{-1}^{2} \frac{t\,dt}{\sqrt{2t^2+8}} = \int_{10}^{16} \frac{\cancel{t}\, du/4\cancel{t}}{\sqrt{u}} = \frac{1}{4} \int_{10}^{16} \frac{1}{\sqrt{u}}\, du = \frac{1}{4} \int_{10}^{16} u^{-1/2}\, du$

$u = 2t^2 + 8 \qquad u(-1) = 10$

$du = 4t\, dt \qquad u(2) = 16$

$dt = du/4t$

$= \frac{1}{4} \left. \frac{u^{1/2}}{1/2} \right|_{10}^{16} = \frac{2}{4} \left(\sqrt{16} - \sqrt{10} \right) = \boxed{\frac{1}{2} \left(4 - \sqrt{10} \right)}$

(4)(a) Part I : If f is a continuous function on $[a,b]$, then the function $F(x) = \int_a^x f(t)\, dt$ is differentiable at all points in (a,b) and:

$$\frac{dF}{dx} = f(x).$$

Part II : Under the same hypothesis as for part I:

$$\int_a^b f(x)\, dx = F(b) - F(a), \quad \left(\text{where } F(x) = \int_a^x f(t)\, dt \right).$$

(b) $$\int_1^x f(t)\,dt = x^2 - 2x + 1$$

differentiate both sides:

$$\frac{d}{dx}\left(\int_1^x f(t)\,dt\right) = \frac{d}{dx}\left(x^2-2x+1\right) = 2x-2$$

From the FTC, we know: $\frac{d}{dx}\left(\int_1^x f(t)\,dt\right) = f(x)$

So: $\boxed{f(x) = 2x-2}$

(5) (a) $$f_{ave} = \frac{1}{3-0}\int_0^3 (x-1)^2\,dx = \frac{1}{3}\int_{-1}^{2} u^2\,du = \frac{1}{3}\,\frac{u^3}{3}\Big|_{-1}^{2} =$$

$u = x-1 \qquad u(0) = -1$

$du = dx \qquad u(3) = 2$

$$= \frac{1}{3}\cdot\left(\frac{8}{3}+\frac{1}{3}\right) = \frac{1}{3}\cdot\frac{9}{3} = 1$$

(b) $f(c) = 1 \iff (c-1)^2 = 1 \iff c-1 = \pm 1 \iff c=0$ or $c=2$

Both values are in $[0,3]$. Conclusion: $\boxed{c=0}$ and $\boxed{c=2}$ are both acceptable values where $f(c) = f_{ave}$.

(6) $$y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$$

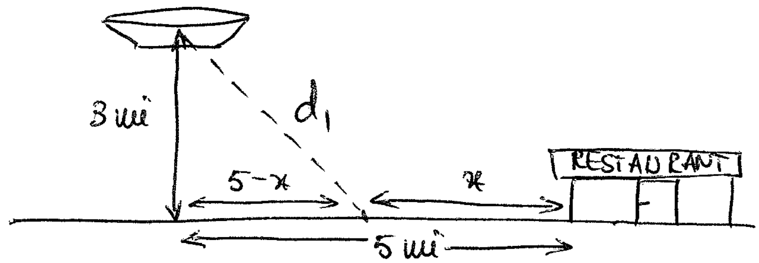
$$\ln y = \ln\left[\sqrt[3]{\frac{x(x-2)}{x^2+1}}\right] = \frac{1}{3}\left[\ln x + \ln(x-2) - \ln(x^2+1)\right]$$

differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+4}\right]$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+4}\right]\cdot\sqrt[3]{\frac{x(x-2)}{x^2+4}}}$$

<u>Extra-credit</u> :



$$d_1 = \sqrt{3^2 + (5-x)^2} \quad \text{and} \quad d_2 = x \qquad (\text{mi})$$

$$T_1 = \frac{d_1}{v} = \frac{\sqrt{9+(5-x)^2}}{v} \quad \text{and} \quad T_2 = \frac{x}{4} \qquad (\text{mi}/\text{h})$$

$$T = T_1 + T_2 = \frac{\sqrt{9+(5-x)^2}}{v} + \frac{x}{4}$$

If v is the rowing speed, then the quickest way to go to the restaurant is given by minimizing T.

$$\frac{dT}{dx} = \frac{1}{2\sqrt{9+(5-x)^2}}\cdot(-2)(5-x)\cdot\frac{1}{v} + \frac{1}{4}$$

we want $\left.\dfrac{dT}{dx}\right|_{x=0} = 0 \implies \dfrac{1}{\cancel{2}\sqrt{9+25}}\cdot(-\cancel{2})\cdot 5\cdot\dfrac{1}{v} + \dfrac{1}{4} = 0$

$$\implies \frac{-5}{\sqrt{34}}\cdot\frac{1}{v} + \frac{1}{4} = 0 \implies \frac{5}{v\sqrt{34}} = \frac{1}{4} \implies v = \boxed{\frac{20}{\sqrt{34}}\ \text{mi}/\text{h}}$$