

1. For  $n = 4$  subintervals,  $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$ . The  $x$ -values are

$$x_0 = 1, \quad x_1 = \frac{5}{4}, \quad x_2 = \frac{3}{2}, \quad x_3 = \frac{7}{4}, \quad x_4 = 2.$$

The corresponding  $y$ -values are

$$y_0 = 1, \quad y_1 = \frac{16}{25}, \quad y_2 = \frac{4}{9}, \quad y_3 = \frac{16}{49}, \quad y_4 = \frac{1}{4}.$$

- a. For 4 rectangles, using left-hand endpoints, the area approximation is

$$\begin{aligned} S_4 &= \frac{1}{4}(y_0 + y_1 + y_2 + y_3) \\ &= \frac{1}{4} \left( 1 + \frac{16}{25} + \frac{4}{9} + \frac{16}{49} \right) \approx .603. \end{aligned}$$

- b. Using the trapezoidal rule for 4 subintervals, the area approximation is

$$\begin{aligned} T_4 &= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{8} \left( 1 + 2 \left( \frac{16}{25} \right) + 2 \left( \frac{4}{9} \right) + 2 \left( \frac{16}{49} \right) + \frac{1}{4} \right) \approx .509. \end{aligned}$$

- c.  $S_4$  is an overestimate because  $f(t) = \frac{1}{t^2}$  is a descending function; using the left-hand endpoints results in rectangles that are larger than the actual area.

$T_4$  is also an overestimate because  $f(t)$  is concave up; each trapezoid is slightly larger than the actual area.

The actual area is

$$\int_1^2 t^{-2} dt = -\frac{1}{t} \Big|_1^2 = -\frac{1}{2} - (-1) = .5.$$

- 2a. The average value of the function is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{1-0} \int_0^1 \sqrt{1-x^2} dx = \int_0^1 \sqrt{1-x^2} dx.$$

We find the value of the definite integral by computing the area of the region between  $f(x)$  and the  $x$ -axis. The graph of  $y = \sqrt{1-x^2}$  on  $[0, 1]$  is a quarter-circle of radius 1. Since the area of a circle is  $\pi r^2$ , the area of the region is  $\pi r^2/4 = \pi(1)^2/4 = \pi/4$ . Therefore the average value of  $f$  on  $[0, 1]$  is  $\pi/4$ .

b. If  $f$  is continuous on  $[a, b]$ , then at some point  $c$  in  $[a, b]$ ,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx = \text{av}(f).$$

c. To find  $c$  such that  $f(c) = \text{av}(f)$ , we set  $f(c)$  equal to  $\pi/4$ :

$$\begin{aligned} f(c) &= \frac{\pi}{4} \\ \sqrt{1-c^2} &= \frac{\pi}{4} \\ 1-c^2 &= \frac{\pi^2}{16} \\ c^2 &= 1 - \frac{\pi^2}{16} \\ c &= \pm \sqrt{1 - \frac{\pi^2}{16}} \\ c &= \frac{\sqrt{16 - \pi^2}}{4}. \end{aligned}$$

There is only one solution  $c$  for the interval  $[0, 1]$ .

3a.

$$\begin{aligned} \int_1^4 \frac{1 + \sqrt{t}}{t} dt &= \int_1^4 \left( \frac{1}{t} + \frac{\sqrt{t}}{t} \right) dt \\ &= \int_1^4 \frac{dt}{t} + \int_1^4 \frac{dt}{\sqrt{t}} \\ &= \ln |t| + 2\sqrt{t} \Big|_1^4 \\ &= \ln 4 + 2\sqrt{4} - (\ln 1 + 2\sqrt{1}) \\ &= \ln 4 + 4 - (0 + 2) \\ &= \ln 4 + 2. \end{aligned}$$

3b. For  $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$ , we let  $u = x^2 + 1$  and  $du = 2x dx$ . The integral then becomes

$$2 \int u^{-1/2} du = 2 \cdot 2u^{1/2} = 4\sqrt{x^2+1} \Big|_0^{\sqrt{3}} = 4(\sqrt{4} - \sqrt{1}) = 4.$$

4a. If  $h(x) = \int_0^{\sqrt{x}} \sin t^2 dt$ , then, by the Fundamental Theorem of Calculus,

$$h'(x) = \frac{d}{dx} \int_0^{\sqrt{x}} \sin t^2 dt = \sin(\sqrt{x})^2 \cdot \frac{1}{2\sqrt{x}} = \frac{\sin x}{2\sqrt{x}}.$$

- 4b. The critical points occur when  $h'(x) = 0$  or is undefined.  $h'(x)$  is undefined at  $x = 0$ .  
 $h'(x) = 0$  when  $\sin x = 0$  at the values  $x = \pm\pi, \pm2\pi, \pm3\pi, \dots$

5. We wish to find

$$s = \int 2 \cos^3 t \sin t \, dt.$$

If we let  $u = \cos t$  and  $du = -\sin t \, dt$ , then the integral becomes

$$s = -2 \int u^3 du = -2 \cdot \frac{u^4}{4} + C = -\frac{1}{2} \cos^4 t + C.$$

We find the value of  $C$  by using the initial value  $s(\pi) = 0$ :

$$0 = -\frac{1}{2} \cos^4(\pi) + C \implies 0 = -\frac{1}{2} + C \implies C = \frac{1}{2}.$$

Therefore  $s = -\frac{1}{2} \cos^4 t + \frac{1}{2}$ .

6. The derivatives of an inverse function pair are reciprocals of each other:

$$\left. \frac{df^{-1}}{dx} \right|_{x=f(a)} = \frac{1}{\left. \frac{df}{dx} \right|_{x=a}}.$$

Since  $\frac{df}{dx} = 3x^2 - 6x$ , then

$$\left. \frac{df^{-1}}{dx} \right|_{x=f(3)} = \frac{1}{3x^2 - 6x \Big|_{x=3}} = \frac{1}{3(3)^2 - 6(3)} = \frac{1}{9}.$$

### Extra Credit Problem

- b. Angle  $\theta$  must lie in the range  $0 \leq \theta \leq \pi/2$  radians.

Since  $\sin \theta = h/10$ , then  $h = 10 \sin \theta$ .

Since  $\cos \theta = x/10$ , then  $x = 10 \cos \theta$ .

Base  $b = 2x + 10 = 20 \cos \theta + 10$ .

- c. The area of the trapezoid is

$$\begin{aligned} A &= \frac{1}{2} h (b_1 + b_2) \\ &= \frac{1}{2} (10 \sin \theta) (20 \cos \theta + 10 + 10) \\ &= (5 \sin \theta) (20 \cos \theta + 20) \\ &= 100 \sin \theta (\cos \theta + 1). \end{aligned}$$

d. We use the product rule to find  $dA/d\theta$ :

$$\begin{aligned}\frac{dA}{d\theta} &= (100 \sin \theta)(-\sin \theta) + (\cos \theta + 1)(100 \cos \theta) \\ &= -100 \sin^2 \theta + 100 \cos^2 \theta + 100 \cos \theta\end{aligned}$$

e. Substituting the identity  $\sin^2 \theta = 1 - \cos^2 \theta$ , we find that

$$\begin{aligned}\frac{dA}{d\theta} &= -100(1 - \cos^2 \theta) + 100 \cos^2 \theta + 100 \cos \theta \\ &= -100 + 100 \cos^2 \theta + 100 \cos^2 \theta + 100 \cos \theta \\ &= 100(2 \cos^2 \theta + \cos \theta - 1) \\ &= 100(2 \cos \theta - 1)(\cos \theta + 1).\end{aligned}$$

We examine the critical points where  $dA/d\theta = 0$ . This occurs when  $2 \cos \theta - 1 = 0$  or  $\cos \theta + 1 = 0$ . Only the first case yields a valid angle  $\theta$  so

$$2 \cos \theta - 1 = 0 \implies \cos \theta = \frac{1}{2} \implies \theta = \frac{\pi}{3} \text{ radians.}$$

The corresponding area is

$$A = 100 \sin \frac{\pi}{3} \left( \cos \frac{\pi}{3} + 1 \right) = 100 \cdot \frac{\sqrt{3}}{2} \cdot \left( \frac{1}{2} + 1 \right) = 75\sqrt{3} \text{ cm}^2.$$

If we check the endpoints,  $\theta = 0$  yields area  $A = 0$ , and  $\theta = \pi/2$  yields area  $A = 100$ , which is less than  $75\sqrt{3}$ .

We verify that  $\theta = \pi/3$  is a local maximum by examining the second derivative  $d^2A/d\theta^2 = -100 \sin \theta(4 \cos \theta + 1)$ , which is negative for  $0 < \theta < \pi/2$ . The function  $A$  is thus concave down, and  $\theta = \pi/3$  does yield the maximum area.