APPM 1350 - Spring 2008 - Final Exam Solutions

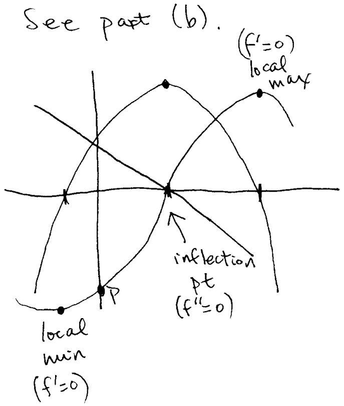
1a. The right-hand and left-hand derivatives must be the same:

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$$

1b. f is continuous everywhere except $x=1$.
g is continuous everywhere.
$(f\circ g)(x) = f(g(x)) = f(x+1) = \frac{1}{x}$
$f\circ g$ is continuous everywhere except $x=0$.

1c. See part (b).

1d.



2a. $$\lim_{x\to-\infty} \frac{3-\frac{2}{x}}{4+\frac{\sqrt{2}}{x^2}} = \lim_{x\to-\infty} \frac{3+\frac{2}{\infty}}{4+\frac{\sqrt{2}}{\infty}} = \frac{3+0}{4+0} = \boxed{\frac{3}{4}}$$

2b. $\lim_{x\to 0} \frac{3^x-1}{2^x-1} = \frac{\text{"0"}}{\text{"0"}}$. Use L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{3^x(\ln 3)}{2^x(\ln 2)} = \boxed{\frac{\ln 3}{\ln 2}}$$

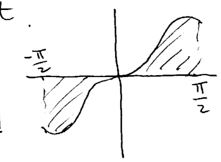
2c. $\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x} = \frac{\text{"0"}}{\text{"0"}}$. Use L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{2x-2}{2x-\cos x} = \frac{0-2}{0-1} = \boxed{2}$$

3a. Solution 1: Since $\sin^5(t) = -\sin^5(-t)$, $\sin^5(t)$ is an odd function, symmetric about $(0,0)$.

$$\int_{-\frac{\pi}{2}}^{0} \sin^5(t)\,dt = -\int_0^{\pi/2} \sin^5(t)\,dt.$$

Therefore $\int_{-\frac{\pi}{2}}^{\pi/2} \sin^5(t)\,dt = \boxed{0}$



Solution 2: Since $\sin^2 t + \cos^2 t = 1$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5(t)\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t)^2(\sin t)\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 t)^2(\sin t)\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-2\cos^2 t+\cos^4 t)(\sin t)\,dt$$

$$= \int(\sin t)\,dt - \int 2\cos^2 t(\sin t)\,dt + \int \cos^4 t(\sin t)\,dt$$

$$= -\cos t + \frac{2\cos^3 t}{3} - \frac{\cos^5 t}{5}\Big]_{-\frac{\pi}{2}}^{\pi/2} = 0-0 = \boxed{0}$$

3b. $$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C$$
$$= \boxed{2\sin\sqrt{x}+C}$$

let $u=\sqrt{x}$
$du = \frac{1}{2\sqrt{x}}dx$

3c. $$\int_0^{2\ln 2}(e^{x/2}-e^{-x/2})\,dx = \int_0^{2\ln 2} e^{x/2}\,dx - \int_0^{2\ln 2} e^{-x/2}\,dx$$

let $u = x/2$, $du = \frac{1}{2}dx$ let $v = \frac{-x}{2}$, $dv = -\frac{1}{2}dx$

$$= 2\int e^u\,du + 2\int e^v\,dv = 2e^u + 2e^v = 2e^{x/2}+2e^{-x/2}\Big]_0^{2\ln 2}$$

$$= 2\cdot 2 + 2\cdot\frac{1}{2} - (2\cdot 1 + 2\cdot 1) = \boxed{1}$$

4a. Logarithmic differentiation:

$$y = \left(\frac{1}{x}+1\right)^x$$

$$\ln y = \ln\left(\frac{1}{x}+1\right)^x = x\ln\left(\frac{x+1}{x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \cancel{x}\left(\frac{x}{x+1}\right)\left(\frac{-1}{\cancel{x^2}}\right) + \ln\left(\frac{x+1}{x}\right)$$

$$\frac{dy}{dx} = \boxed{\left(\frac{1}{x}+1\right)^x\left[\frac{-1}{x+1} + \ln\left(\frac{1}{x}+1\right)\right]}$$

4b. $\lim\limits_{x\to\infty}\left(\frac{1}{x}+1\right)^x = \text{"}1^\infty\text{"}$ We use L'Hôpital's Rule for indeterminate powers.

let $L = \lim\limits_{x\to\infty} \ln\left(\frac{1}{x}+1\right)^x$

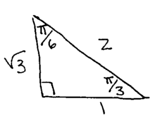
$$= \lim_{x\to\infty} x\ln\left(\frac{x+1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{\left(\frac{x}{x+1}\right)\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\frac{-1}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{1}{1} = 1$$

$$\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^x = e^L = e^1 = \boxed{e}$$

5a. $f(-\sqrt{3}) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.



30-60-90 triangle

Since $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

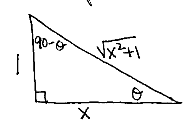
and the range of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \boxed{-\frac{\pi}{6}}$$

$$g(\sqrt{3}) = \cos^{-1}\frac{1}{\sqrt{3+1}} = \cos^{-1}\frac{1}{2} = \boxed{\frac{\pi}{3}}$$ (see diagram)

5b. $-f(x)$ and $g(x)$ are complementary angles that sum to $90°$:

$$g(x) - f(x) = \frac{\pi}{2}$$



$\tan(-\theta) = \frac{-1}{x}$

$\cos(90-\theta) = \frac{1}{\sqrt{x^2+1}}$

5c. $L(x) = f(a) + f'(a)(x-a)$, $a = 1$

$$f(x) = \tan^{-1}\frac{1}{x} \Rightarrow f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(\frac{-1}{x^2}\right) = \frac{-1}{x^2+1}$$

$$f(1) = \tan^{-1} 1 = \frac{\pi}{4}, \quad f'(1) = \frac{-1}{1+1} = \frac{-1}{2}$$

$$L(x) = f(1) + f'(1)(x-1)$$

$$\boxed{L(x) = \frac{\pi}{4} - \frac{1}{2}(x-1)}$$

6. Law of exponential change: $y = y_0 e^{-kt}$

After 1 year, 90% of y_0 is left:

$$.9y_0 = y_0 e^{-k(1)}$$

$$.9 = e^{-k}$$

$$\ln .9 = -k$$

$$\boxed{k = -\ln .9 \approx .10536}$$

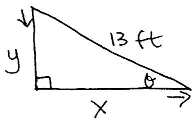
Find t when oil level is 20% of y_0.

$$.2y_0 = y_0 e^{(\ln .9)t}$$

$$.2 = (.9)^t$$

$$\ln .2 = \ln .9^t = t\ln .9$$

$$\boxed{t = \frac{\ln .2}{\ln .9} \approx 15.28 \text{ years}}$$

7. 

When $x = 12$, $\frac{dx}{dt} = 5$ ft/sec

$x^2 + y^2 = 13^2$

when $x = 12$ ft, $y = \sqrt{13^2 - 12^2} = 5$ ft.

a. Find $\frac{dy}{dt}$ when $x = 12$ ft and $y = 5$ ft

$$x^2 + y^2 = 169$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$
$$(12)(5) + (5)\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = \frac{-12(5)}{5} = \boxed{-12 \text{ ft/sec}}$$

b. Find $\frac{dA}{dt}$ where $A = \frac{1}{2}xy$

$$A = \frac{1}{2}xy$$
$$\frac{dA}{dt} = \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right] = \frac{1}{2}\left((12)(-12) + (5)(5)\right)$$
$$= \frac{1}{2}(-144 + 25) = \boxed{\frac{-119}{2} \text{ ft}^2/\text{sec}}$$

c. We can use any of the basic trig functions to find $\frac{d\theta}{dt}$.

$$\cos\theta = \frac{x}{13}$$
$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$$

When $x = 12$, $\sin\theta = \frac{5}{13}$.

$$-\left(\frac{5}{13}\right)\frac{d\theta}{dt} = \frac{1}{13}(5)$$
$$\boxed{\frac{d\theta}{dt} = -1 \text{ radian/sec}}$$

8. We wish to maximize v when $r = 1$.

$$v = x^2\ln\left(\frac{1}{x}\right)$$
$$\frac{dv}{dx} = x^2(x)\left(\frac{-1}{x^2}\right) + 2x\ln\left(\frac{1}{x}\right)$$
$$= -x + 2x\ln\left(\frac{1}{x}\right)$$

We set $\frac{dv}{dx} = 0$.

$$-x + 2x\ln\left(\frac{1}{x}\right) = 0$$
$$2x\ln\left(\frac{1}{x}\right) = x$$
$$\ln\left(\frac{1}{x}\right) = \frac{1}{2}$$
$$-\ln x = \frac{1}{2}$$
$$\ln x = -\frac{1}{2}$$
$$x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

Since $x = \frac{r}{h} = \frac{1}{h}$,

$$\boxed{h = \sqrt{e}}$$

We check that $x = \frac{1}{\sqrt{e}}$ is a max point by examining the second derivative.

$$\frac{d^2v}{dx^2} = -1 + 2x(x)\left(-\frac{1}{x^2}\right) + 2\ln\left(\frac{1}{x}\right)$$
$$= -3 + 2\ln\left(\frac{1}{x}\right)$$

When $x = \frac{1}{\sqrt{e}}$, $\frac{d^2v}{dx^2} = -3 + 2\left(\frac{1}{2}\right) = -2$.

Therefore the function is concave down and $x = \frac{1}{\sqrt{e}}$ is indeed a max point.