APPM 1350 Summer 2008
Final Exam Solutions

1. a) $\int \sqrt{\theta} + e^{2\theta} + \pi \, d\theta$

$= \frac{2}{3}\theta^{3/2} + \frac{1}{2}e^{2\theta} + \pi\theta + C$

b) $\int_0^2 \frac{x}{x^2-5}\,dx$ $\quad u = x^2-5 \quad du = 2x\,dx$

$u(0) = -5 \quad u(2) = -1$

$= \frac{1}{2}\int_{-5}^{-1} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{-5}^{-1}$

$= \frac{1}{2}(\ln|-1| - \ln|-5|$

$= -\frac{1}{2}\ln 5 \boxed{= \ln 5^{-1/2}}$

c) $\int \sin^2(r)\,dr = \int \frac{1-\cos 2r}{2}\,dr \boxed{= \frac{1}{2}r - \frac{1}{4}\sin 2r + C}$

d) $\int \frac{z}{\sqrt{z^2+9}}\,dz$ $\quad u = z^2+9 \quad du = 2z\,dz$

$= \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C$

$\boxed{= (z^2+9)^{1/2} + C}$

e) $\int_0^1 \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$ $\quad u = \arcsin(x) \quad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$u(0) = 0 \quad u(1) = \pi/2$

$= \int_0^{\pi/2} u\,du = \frac{1}{2}u^2\Big|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi^2}{4} - 0\right) \boxed{= \frac{\pi^2}{8}}$

f) $\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\,d\theta$ $\quad u = \cos\theta \quad du = -\sin\theta\,d\theta$

$= -\int \frac{1}{1+u^2}\,du = -\tan^{-1}u + C$

$\boxed{= -\tan^{-1}(\cos\theta) + C}$

2. a) $y = \ln(3xe^{2x}(2x-1)^6)$

$$\boxed{y' = \frac{3e^{2x}(2x-1)^6 + 6xe^{2x}(2x-1)^6 + 36xe^{2x}(2x-1)^5}{3xe^{2x}(2x-1)^6}}$$ (direct method)

or

$y = \ln 3 + \ln x + \ln e^{2x} + 6\ln(2x-1)$

$$\boxed{y' = 0 + \frac{1}{x} + 2 + \frac{12}{2x-1}}$$

b) $y = \pi^x + \log_3(x^5) = \pi^x + \frac{5\ln x}{\ln 3}$

$$\boxed{y' = \pi^x \ln \pi + \frac{5}{\ln 3}\,\frac{1}{x}}$$

c) $y = x\sin^{-1}(x) + \sqrt{1-x^2}$

$$y' = \sin^{-1}(x) + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{1}{2}\left(\frac{-2x}{\sqrt{1-x^2}}\right) \boxed{= \sin^{-1}(x)}$$

d) $y = \frac{\csc(x)}{2x+0.5}$

$$\boxed{y' = \frac{(2x+0.5)(-\csc x \cot x) - 2\csc x}{(2x+0.5)^2}}$$

e) $y = e^{xy} \Rightarrow \ln y = xy$

$$\frac{y'}{y} = y + xy'$$

$$y' = y^2 + xyy'$$

$$y' - xyy' = y^2 \Rightarrow \boxed{y' = \frac{y^2}{1-xy}}$$

3. a) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x-1} = \dfrac{0}{0}$

$\Rightarrow \lim\limits_{x\to 1} \dfrac{2x+1}{1} \boxed{= 3}$

b) $\lim\limits_{x\to\infty} \dfrac{7844x^{13}+400x^{12}}{0.000005x^{14}} \boxed{= 0}$

(denominator has higher degree than numerator)

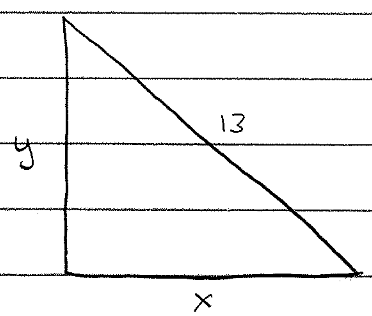
c) $\lim\limits_{x\to 0} \dfrac{x-\sin(x)}{x^3} = \dfrac{0}{0}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} = \dfrac{0}{0}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{\sin x}{6x} = \dfrac{0}{0}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{\cos x}{6} \boxed{= \dfrac{1}{6}}$

4.



$x^2+y^2 = 13^2$

$\left.\dfrac{dx}{dt}\right|_{x=12} = 10$

find $\dfrac{dy}{dt}$

$x^2+y^2 = 13^2$

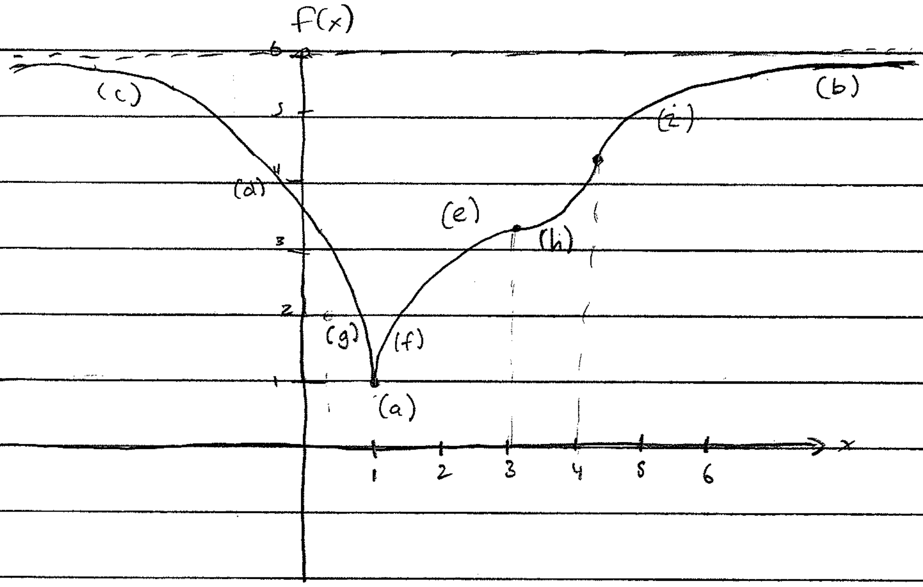
$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad \dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt} = -\dfrac{12}{y}\cdot 10$

find y when $x = 12$, $y^2 = 13^2 - 12^2 = 169 - 144 = 25$

$y = 5$

so $\dfrac{dy}{dt} = -\dfrac{12}{5}\cdot 10 \boxed{= -24 \text{ ft/sec}}$

5.



6. a) $\frac{dT}{dt} = k(T - T_s)$

b) $\int \frac{dT}{T-T_s} = \int k\,dt \Rightarrow \ln|T-T_s| = kt + c$

$$\Rightarrow T - T_s = ce^{kt}$$

$$T = ce^{kt} + T_s$$

$T(0) = c + T_s = T_0 \Rightarrow c = (T_0 - T_s)$

$T(t) = (T_0 - T_s)e^{kt} + T_s$

c) $T_0 = 130°F \qquad T_s = 80°F$

$T(10) = 120°F = (130-80)e^{10k} + 80$

$4 = 5e^{10k}$

$\frac{4}{5} = e^{10k} \Rightarrow \frac{1}{10}\ln\left(\frac{4}{5}\right) = k$ units of $\frac{1}{min}$

d) $T(-20) = (130-80)e^{-20k} + 80$ (k defined from part c)

7. $V(r) = c r^2(r_0 - r) = c r^2 r_0 - c r^3$

a) $\frac{dV}{dr} = 2 c r r_0 - 3 c r^2$

$r(2 c r_0 - 3 c r) = 0 \quad \Rightarrow r = 0 \quad$ or $\quad 2 c r_0 - 3 c r = 0$

$\Rightarrow 3 \cancel{c} r = 2 \cancel{c} r_0 \qquad r = \frac{2}{3} r_0$ is a critical point

b) $\frac{d^2 V}{dr^2} = 2 c r_0 - 6 c r = 2(2)(3) - 6(2)\left(\frac{2}{3}(3)\right)$

$= 12 - 24 = {}^-12 < 0$ so

$r = \frac{2}{3} r_0$ is a local max

c) since $\frac{2}{3} r_0 < r_0$ the windpipe constricts when you cough

since $r = \frac{2}{3} r_0$ is a max for $V(r)$, this means the velocity of windflow increases