APPM 1350

FALL 2008, EXAM 3

① a) (HW: 6.3 #10, pg 472)

$$\ln(y^2-1) - \ln(y+1) = \ln(\sin x)$$

$$\ln[(y+1)(y-1)] - \ln(y+1) = \ln(\sin x)$$

$$\ln(y+1) + \ln(y-1) - \ln(y+1) = \ln(\sin x)$$

$$\ln(y-1) = \ln(\sin x)$$

$$y-1 = \sin x$$

$$\boxed{y = \sin x + 1}$$

Note: Orig. Eq Is only valid when $\sin x > 0$, so domain of y is <u>not</u> all real numbers

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b) MVT: If $f(x)$ is continuous on $[a,b]$ then there is a c on $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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c) $f(x) = e^{-x} - 3x - \sin x$, so

$$f'(x) = -e^{-x} - 3 - \cos x$$

$-1 \le -\cos x \le 1$, so

$$-4 \le -3 - \cos x \le -2$$

Since $-e^{-x} < 0$, $-e^{-x} - 3 - \cos x < -2$

$\Rightarrow f'(x) < 0$, so $f(x)$ is a decreasing function. Since $f(x)$ is decreasing, it is one-to-one, and since all one-to-one functions have inverses,

$$\boxed{\text{yes, } f(x) \text{ is invertible}}$$

② a) Want to estimate $\int_1^4 \frac{1}{1+x^2}\,dx$ using a Riemann Sum with 3 equally spaced subintervals and left-hand endpoints.

Riemann Sum: $\sum_{k=1}^{n} f(c_k)\Delta x_k$

3 intervals $\Rightarrow n = 3$

equally spaced $\Rightarrow \Delta x_k = \frac{4-1}{3} = 1$ for all k

So our subintervals are $[1,2]$, $[2,3]$, and $[3,4]$.

Using left-hand endpoints, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$

$$\Rightarrow \sum_{k=1}^{3} f(c_k)\Delta x_k = f(1)(1) + f(2)(1) + f(3)(1)$$

$$= \frac{1}{1^2+1}(1) + \frac{1}{2^2+1}(1) + \frac{1}{3^2+1}(1)$$

$$= \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$$

$$= \frac{5}{10} + \frac{2}{10} + \frac{1}{10} = \frac{8}{10} = \boxed{\frac{4}{5}}$$

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b) Since $f(x) = \frac{1}{x^2+1}$, $f'(x) = \frac{-2x}{(x^2+1)^2}$.

Note that $f'(x) < 0$ if $x > 0$, so $f'(x) < 0$ on $[1,4]$. Therefore $f(x)$ is decreasing over the interval $[1,4]$, so the left-hand Riemann Sum will <u>over-estimate</u> the integral.

③ a) (HW: 6.2 #59, pg 466)

$$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} \frac{du}{u^2}$$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$\int_{\ln 2}^{\ln 4} \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln 2}^{\ln 4}$$

$$= \frac{-1}{\ln 4} - \frac{-1}{\ln 2}$$

$$= \frac{-1}{2\ln 2} + \frac{2}{2\ln 2}$$

$$= \boxed{\frac{1}{2\ln 2}}$$

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b) $\int e^t \sin(e^t - 2)\,dt = \int \sin(u)\,du$

$u = e^t - 2 \Rightarrow du = e^t\,dt$

$$\int \sin(u)\,du = -\cos u + C$$

$$= \boxed{-\cos(e^t - 2) + C}$$

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c) $\int_3^{-1}(1 + f(x))\,dx = -1$

$$\int_{-1}^{3}(1 + f(x))\,dx = 1$$

$$\int_{-1}^{3} dx + \int_{-1}^{3} f(x)\,dx = 1$$

$$4 + \int_{-1}^{3} f(x)\,dx = 1$$

$$\int_{-1}^{3} f(x)\,dx = -3$$

$\int_{-1}^{5}\left(2f(x) - \frac{1}{6}\right)dx = 11$

$$\int_{-1}^{5} 2f(x)\,dx - \int_{-1}^{5}\frac{1}{6}\,dx = 1$$

$$2\int_{-1}^{5} f(x)\,dx - 1 = 1$$

$$2\int_{-1}^{5} f(x)\,dx = 12$$

$$\int_{-1}^{5} f(x)\,dx = 6$$

$$\int_3^5 f(x)\,dx = \int_{-1}^{5} f(x)\,dx - \int_{-1}^{3} f(x)\,dx$$

$$= 6 - (-3) = \boxed{9}$$

4 $g(x) = 3 + \int_1^{x^2} (1+\ln t)\,dt$

a) For linearization at $x=-1$, need to find a line with slope $g'(-1)$ that passes through the point $(-1, g(-1))$.

$$g(-1) = 3 + \int_1^{(-1)^2} (1+\ln t)\,dt$$
$$= 3 + \int_1^1 (1+\ln t)\,dt$$
$$= 3 + 0 = 3$$

$$g'(x) = \frac{d}{dx}[3] + \frac{d}{dx}\int_{-1}^{x^2} (1+\ln t)\,dt$$
$$= 0 + (1+\ln x^2)\frac{d}{dx}[x^2]$$
$$= (1+\ln x^2)2x$$

$$g'(-1) = (1+\ln(-1)^2)2(-1)$$
$$= (1+\ln 1)(-2)$$
$$= (1+0)(-2) = -2$$

So $L(x)$ is a line w/ slope $m=-2$ that passes through the point $(-1, 3)$. In point-slope form:

$$L(x) - 3 = -2(x-(-1))$$
$$L(x) = -2(x+1)+3 = \boxed{-2x+1}$$

b) $g'(x) = (1+\ln(x^2))2x$ (from above)

So
$$g''(x) = \left(\frac{1}{x^2}\cdot 2x\right)2x + (1+\ln(x^2))2$$
$$= 4 + 2(1+\ln(x^2))$$
$$= 6 + 2\ln(x^2)$$

$$g''(-1) = 6 + 2\ln(-1)^2$$
$$= 6 + 2\ln 1 = \boxed{6}$$

5 (HW: 4.2 #32a, pg 289)

i) $\frac{d^2y}{dx^2} = 6x$

ii) y passes through $(0,1)$ with horizontal tangent

$\Rightarrow$ $y(0)=1$ (passes through point)

and $y'(0)=0$ (horizontal tangent)

Since $y''(x) = 6x$ and $y'(0)=0$, we can find $y'(x)$:

$$y'(x) = \int 6x\,dx = 3x^2 + C$$
$$y'(0) = 3(0^2) + C = 0$$
$$\Rightarrow C = 0$$

So $y'(x) = 3x^2$. Combined with $y(0)=1$, find $y(x)$:

$$y(x) = \int 3x^2\,dx = x^3 + C$$
$$y(0) = 0^3 + C = 1$$
$$\Rightarrow C = 1$$

So $\boxed{y(x) = x^3+1}$

6 $y = x^2 - 3x + 2$
$= (x-2)(x-1)$

So y has roots ~~on~~ at $x=1$ & $x=2$.

- On $[-1,1)$, both $(x-2)$ and $(x-1)$ are negative, so $y=(x-2)(x-1)$ is positive.
- On $(1,2)$, $(x-2)$ is negative but $(x-1)$ is positive, so $y=(x-2)(x-1)$ is negative.

Therefore the total area between $y(x)$ and the x-axis on the interval $[-1,2]$ is

$$A = \int_{-1}^{1}(x^2-3x+2)\,dx - \int_1^2 (x^2-3x+2)\,dx$$
$$= \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Big|_{-1}^{1} - \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Big|_1^2$$
$$= \left[\left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(-\frac{1}{3} - \frac{3}{2} - 2\right)\right] - \left[\left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right]$$
$$= \left[\left(\frac{2}{6} - \frac{9}{6} + \frac{12}{6}\right) + \left(\frac{2}{6} + \frac{9}{6} + \frac{12}{6}\right)\right] - \left[\left(\frac{8}{3} - 2\right) - \left(\frac{2}{6} - \frac{9}{6} + \frac{12}{6}\right)\right]$$
$$= \left(\frac{5}{6} + \frac{23}{6}\right) - \left(\frac{2}{3} - \frac{5}{6}\right)$$
$$= \frac{28}{6} - \left(-\frac{1}{6}\right) = \boxed{\frac{29}{6}}$$