APPM 1350 FALL 2008

ALTERNATE FINAL SOLUTIONS

1 a) $\int \frac{dy}{y^2+6y+10} = \int \frac{dy}{(y+3)^2+1}$ (complete the square)

$u = y+3 \Rightarrow du = dy$

$$\int \frac{dy}{(y+3)^2+1} = \int \frac{du}{u^2+1}$$

$$= \tan^{-1} u + C$$

$$\boxed{= \tan^{-1}(y+3) + C}$$

b) $\int_0^{\frac{3\sqrt{2}}{4}} \frac{ds}{\sqrt{9-4s^2}} = \frac{1}{2}\int_0^{\frac{3\sqrt{2}}{2}} \frac{du}{\sqrt{9-u^2}}$

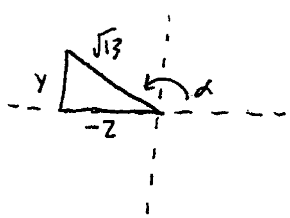
$u = 2s \Rightarrow du = 2ds$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{3}\right)\Big|_0^{\frac{3\sqrt{2}}{2}}$$

$$= \frac{1}{2}\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1} 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{8}}$$

c) $\alpha = \sec^{-1}\left(\frac{-\sqrt{13}}{2}\right)$ $\quad \frac{-\sqrt{13}}{2} < 0 \Rightarrow \alpha \in \left(\frac{\pi}{2}, \pi\right]$ (second quadrant)



note that $y > 0$ in 2nd quad

and $y^2 + (-2)^2 = \sqrt{13}^2$

$\Rightarrow y^2 = 13 - 4 = 9$

$y = 3$ (since $y > 0$)

$\sin\alpha = \frac{y}{\sqrt{13}} = \boxed{\frac{3}{\sqrt{13}}}$

$\cos\alpha = \frac{1}{\sec\alpha} = \boxed{\frac{-2}{\sqrt{13}}}$

$\tan\alpha = \frac{y}{-2} = \boxed{-\frac{3}{2}}$

2 a) $\lim_{x\to\infty} \frac{1}{x\ln x}\int_1^x t^{-1}\,dt = \lim_{x\to\infty} \frac{\ln x}{x\ln x} = \lim_{x\to\infty} \frac{1}{x} = \boxed{0}$

b) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = L$, so $\ln L = \ln \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

$\ln L = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

L'Hôpital: $\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$

then $\ln L = 1 \Rightarrow L = e^1 = \boxed{e}$

c) $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x = L$, so $\ln L = \ln \lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

$\ln L = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$

L'Hôpital: $\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$

$\ln L = 2 \Rightarrow L = \boxed{e^2}$

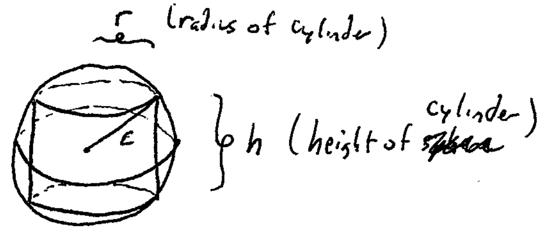
d) $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \lim_{x\to\infty} \sqrt{\frac{9x+1}{x+1}} = \sqrt{\lim_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{9} = \boxed{3}$

e) $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 x} = \lim_{x\to\infty} \frac{\ln(x+1)}{\left(\frac{\ln x}{\ln 2}\right)} = \ln 2\cdot\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x}$

L'Hôpital: $\ln 2\cdot\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \ln 2 \lim_{x\to\infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)}$

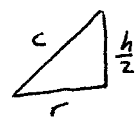
$$= \ln 2 \lim_{x\to\infty} \frac{x}{x+1} = \boxed{\ln 2}$$

(3)



To inscribe cylinder in a sphere of radius $\sqrt{3}$, need $r^2 + \left(\frac{h}{2}\right)^2 = (\sqrt{3})^2 \Rightarrow r^2 = 3 - \frac{h^2}{4}$

(The segment labeled "c" in the above diagram is both the radius of the sphere and hypotenus of the triangle at right:



)

Want to maximize V, volume of cylinder:

$$V = \pi r^2 h = \pi\left(3 - \frac{h^2}{4}\right)h = \frac{\pi}{4}(12h - h^3)$$

Critical points:

$$\frac{dV}{dh} = \frac{\pi}{4}(12 - 3h^2) = \frac{3\pi}{4}(4 - h^2)$$

So $\frac{dV}{dh} = 0$ when $h^2 - 4 = 0 \Rightarrow h = \pm 2$

Note that $h = -2$ does not make sense physically, so $\boxed{h = 2}$

$r^2 = 3 - \frac{2^2}{4} = 3 - 1 = 2 \Rightarrow \boxed{r = \sqrt{2}}$ (again, $r = -\sqrt{2}$ does not make sense). This is a maximum because

$$\frac{d^2V}{dh^2} = \frac{-6\pi h}{4}, \text{ so } \frac{d^2V}{dh^2} = -3\pi < 0 \text{ when } h = 2.$$

$$V_{max} = \pi r^2 h = \pi(\sqrt{2})^2 2 = \boxed{4\pi}$$

(4) a) $\lim_{x\to\infty} \frac{x^x}{e^x} = \lim_{x\to\infty} \left(\frac{x}{e}\right)^x = "\infty^\infty" = \infty$

So x^x grows faster than e^x

$$\lim_{x\to\infty} \frac{e^{x/2}}{e^x} = \lim_{x\to\infty} e^{-x/2} = 0$$

So $e^{x/2}$ grows slower than e^x

$\Rightarrow e^{x/2}$ is slowest, then e^x, then x^x is fastest

b) $\lim_{x\to 0} e^x = e^0 = \boxed{1}$ (e^x is continuous at $x=0$)

$\lim_{x\to 0} e^{x/2} = e^{0/2} = \boxed{1}$ ($e^{x/2}$ is continuous at $x=0$)

$\lim_{x\to 0} x^x = L \Rightarrow \ln L = \lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}$

By L'Hôpital's rule, $\ln L = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{-x^2}{x} = \lim_{x\to 0} -x = 0$

Since ~~the~~ $\ln L = 0$, it follows that $L = e^0 = \boxed{1}$

c) $\frac{d}{dx}[e^x] = \boxed{e^x}$ when $x=1$, derivative $= e^1 = \boxed{e}$

$\frac{d}{dx}[e^{x/2}] = \boxed{\frac{1}{2}e^{x/2}}$ when $x=1$, derivative $= \frac{1}{2}e^{1/2} = \boxed{\frac{1}{2}\sqrt{e}}$

$y = x^x$, want to find $\frac{dy}{dx} = \frac{d}{dx}[x^x]$

$\ln y = \ln x^x = x \ln x \Rightarrow \frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + 1\cdot \ln x = 1 + \ln x$

So $\frac{dy}{dx} = y(1 + \ln x) = \boxed{x^x(1 + \ln x)}$ when $x=1$, derivative is $1^1(1 + \ln 1) = 1(1+0) = \boxed{1}$

d) On last page...

(5) a) avg value $= \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \frac{6}{9+x^2}\,dx$

$$= \frac{6}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{dx}{9+x^2}$$

$$= \frac{6}{\sqrt{3}}\cdot\left(\frac{1}{3}\tan^{-1}\frac{x}{3}\Big|_0^{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}-0\right) = \boxed{\frac{\pi}{3\sqrt{3}}}$$

b) avg value $= \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \frac{6x\,dx}{9+x^2}$

$u = 9+x^2 \Rightarrow du = 2x\,dx$

$$= \frac{3}{\sqrt{3}}\int_9^{12}\frac{du}{u}$$

$$= \frac{3}{\sqrt{3}}\ln u\Big|_9^{12}$$

$$= \frac{3}{\sqrt{3}}(\ln 12 - \ln 9) = \frac{3}{\sqrt{3}}\ln\frac{12}{9} = \boxed{\frac{3}{\sqrt{3}}\ln\left(\frac{4}{3}\right)}$$

c) tot area $= \int_{1/e}^{e}\left|\frac{\ln x}{x}\right|dx$

$\frac{\ln x}{x} \le 0$ for $\frac{1}{e} \le x \le 1$

$\frac{\ln x}{x} \ge 0$ for $1 \le x \le e$

$\Rightarrow$ tot area $= -\int_{1/e}^{1}\frac{\ln x}{x}dx + \int_1^e \frac{\ln x}{x}dx$

let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

tot area $= -\int_{-1}^{0} u\,du + \int_0^1 u\,du$

$$= -\frac{u^2}{2}\Big|_{-1}^{0} + \frac{u^2}{2}\Big|_0^1 = -\left(0-\frac{1}{2}\right) + \left(\frac{1}{2}-0\right)$$

$$= \frac{1}{2}+\frac{1}{2} = \boxed{1}$$

(6) From picture, let x = distance from car to point directly in front of camera

a) $\tan\theta = \frac{x}{132\text{ ft}} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{132\text{ ft}}\,\frac{dx}{dt}$

when car is directly in front of you, $\theta = 0 \Rightarrow \sec^2\theta = 1$

So $\frac{d\theta}{dt} = \frac{1}{132\text{ ft}}\frac{dx}{dt} = \frac{1}{132\text{ ft}}(\pm 264\text{ ft/sec})$

x is getting smaller as car approaches and bigger as car continues to move

$\frac{d\theta}{dt} = 2\text{ ~/sec} = \boxed{2\text{ rad/sec} \;\; \text{or} \;\; -2\text{ rad/sec}}$ (either sign on $\frac{dx}{dt}$ is okay)

$\frac{1}{2}$ second later, $x = 132$ ft (since $\frac{dx}{dt} = 264$ ft/sec) $\Rightarrow \theta = \frac{\pi}{4}$.

So $\sec^2\frac{\pi}{4}\,\frac{d\theta}{dt} = \frac{1}{132\text{ ft}}\cdot 264\text{ ft/sec}$ (now $\frac{dx}{dt}$ <u>must</u> be >0)

$2\frac{d\theta}{dt} = 2\text{ rad/sec} \Rightarrow \frac{d\theta}{dt} = \boxed{1\text{ rad/sec}}$

b) From above, $\tan\theta = \frac{x}{132\text{ ft}} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{x}{132\text{ ft}}\right)}$

c) $\frac{d\theta}{dt} = \frac{d}{dt}\left[\tan^{-1}\left(\frac{x}{132\text{ ft}}\right)\right]$

$$= \frac{1}{1+\left(\frac{x}{132\text{ ft}}\right)^2}\cdot\frac{1}{132\text{ ft}}\cdot\frac{dx}{dt}$$

$$= \boxed{\frac{132\text{ ft}}{(132\text{ ft})^2+x^2}\cdot\frac{dx}{dt}}$$

note that, when $x=0$, $\frac{d\theta}{dt} = 2$ rad/sec and, when $x = 132$ ft, $\frac{d\theta}{dt} = 1$ rad/sec (just like in part a)

7 a) $\ln 2 = \int_1^2 \frac{1}{t}\,dt$

Trap Rule: $\int_a^b f(t)\,dt \approx \frac{h}{2}\left(f(a) + 2f(t_1) + \dots + 2f(t_{n-1}) + f(b)\right)$,

where $h = \frac{b-a}{n}$; $t_k = a + kh$

For this problem $n = 3 \Rightarrow h = \frac{2-1}{3} = \frac{1}{3}$

So $t_1 = 1 + \frac{1}{3} = \frac{4}{3}$; $t_2 = 1 + \frac{2}{3} = \frac{5}{3}$

$$\int_1^2 \frac{1}{t}\,dt \approx \frac{\frac{1}{3}}{2}\left(\frac{1}{1} + 2\left(\frac{1}{(\frac{4}{3})}\right) + 2\left(\frac{1}{(\frac{5}{3})}\right) + \frac{1}{2}\right)$$

$$= \frac{1}{6}\left(1 + \frac{6}{4} + \frac{6}{5} + \frac{1}{2}\right)$$

$$= \frac{1}{6}\left(\frac{20 + 30 + 24 + 10}{20}\right)$$

$$= \frac{1}{6}\left(\frac{84}{20}\right) = \frac{14}{20} = \boxed{\frac{7}{10}}$$

b) $M = \max |f''(t)|$ on $[a,b]$

$f(t) = \frac{1}{t} \Rightarrow f'(t) = \frac{-1}{t^2}$, $f''(t) = \frac{2}{t^3}$. On $[1,2]$, $\left|\frac{2}{t^3}\right| \le 2$

$\Rightarrow M = 2$. So $|E_T| \le \frac{2-1}{12} \cdot \left(\frac{1}{3}\right)^2 \cdot 2 = \frac{1}{12} \cdot \frac{1}{9} \cdot 2 = \boxed{\frac{1}{54}}$

c) $|E_T| \le \frac{2-1}{12} \cdot \left(\frac{1}{n}\right)^2 \cdot 2 \le \frac{1}{6} \cdot 10^{-6}$

$\frac{1}{6n^2} \le \frac{1}{6} \cdot 10^{-6} \Rightarrow n^2 \ge 10^6$

So $\boxed{n \ge 10^3} = 1000$ results in error less than $\frac{1}{6} \cdot 10^{-6}$

d) $e^y = 2 \Rightarrow e^y - 2 = 0$. Want root of $f(y) = e^y - 2$:

Newton's Method: $y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$

$f(y) = e^y - 2 \Rightarrow f'(y) = e^y$

So $y_{n+1} = y_n - \frac{e^{y_n} - 2}{e^{y_n}}$

$= y_n - 1 + \frac{2}{e^{y_n}}$

$y_0 = 0$

$y_1 = 0 - 1 + \frac{2}{e^0}$

$= -1 + 2 = \boxed{1}$

$y_2 = 1 - 1 + \frac{2}{e^1}$

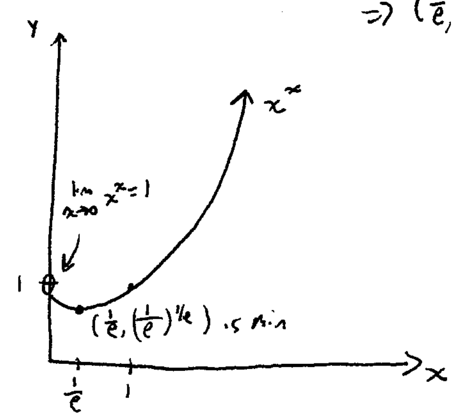
$= \boxed{\frac{2}{e}}$

9 d) From part (c), $\frac{d}{dx}[x^x] = x^x(\ln x + 1)$, so critpoint @ $x = \frac{1}{e}$

$$\frac{d^2}{dx^2}[x^x] = \frac{d}{dx}[x^x](\ln x + 1) + x^x\left(\frac{1}{x}\right)$$

$= x^x\left[(\ln x + 1)^2 + \frac{1}{x}\right] > 0$, so x^x is concave up for $x > 0$.

$\Rightarrow \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$ is a minimum.



There are no maximums or asymptotes