 APPM 1350 - Spring 2009

Midterm 2 - solutions

(1) (a) Consider the function:

$$f(x) = x - \frac{1}{x^2+1} \quad \text{continuous at all reals}$$

$$f(0) = 0 - \frac{1}{0+1} = -1 < 0$$

$$f(1) = 1 - \frac{1}{1+1} = \frac{1}{2} > 0$$

By the Intermediate Value Theorem, the function has a root in $(0,1)$

$\Rightarrow$ the equation has at least one solution in $(0,1) \Rightarrow$ $\boxed{\text{TRUE}}$

(b) $\boxed{\text{FALSE}}$: Counterexample: If $f(x) = g(x) + c$, where c is any constant $\neq 0$, then $f' = g'$, but $f \neq g$.

(c) $\boxed{\text{TRUE}}$: $f' > 0 \Rightarrow f$ increasing.

(2) (a) $$\lim_{x\to\infty} \frac{x^2+x}{5x^2+1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{x}{x^2}}{\frac{5x^2}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{5+\frac{1}{x^2}} = \frac{1}{5}$$

(dominance of powers criterion)

$\lim_{x\to\infty} \frac{\sin x}{5x^2+1} = 0$, by the Sandwich Theorem. Indeed:

$$-1 \le \sin x \le 1$$

$$\frac{-1}{5x^2+1} \le \frac{\sin x}{5x^2+1} \le \frac{1}{5x^2+1}$$

when $x \to \infty$

$$0$$

In conclusion:

$$\lim_{x\to\infty} \frac{x^2+x+\sin x}{5x^2+1} = \lim_{x\to\infty} \frac{x^2+x}{5x^2+1} + \lim_{x\to\infty} \frac{\sin x}{5x^2+1} =$$

$$= \frac{1}{5} + 0 = \boxed{\frac{1}{5}}$$

(b) $x^2 = \dfrac{x-y}{x+y}$

Implicit differentiation (differentiate both sides w.r.t x)

$$\frac{d}{dx}(x^2) = \frac{d}{dx}\left[\frac{x-y}{x+y}\right] \quad \text{(use quotient rule on the right)}$$

$$2x = \frac{(1-dy/dx)(x+y) - (1+dy/dx)(x-y)}{(x+y)^2} =$$

$$= \frac{(\cancel{x}+y-\cancel{x}+y) - dy/dx\,(x+\cancel{y}+x-\cancel{y})}{(x+y)^2} = \frac{2y-2x\frac{dy}{dx}}{(x+y)^2}$$

$$\Rightarrow \quad 2y - 2x\frac{dy}{dx} = 2x(x+y)^2$$

$$-2x\frac{dy}{dx} = 2x(x+y)^2 - 2y$$

$$\frac{dy}{dx} = \frac{2y-2x(x+y)^2}{2x}$$

At the point $(1,0)$: $\left.\dfrac{dy}{dx}\right|_{(1,0)} = \dfrac{0-2}{2} = -1$

Equation of tangent line : slope $m=-1$, point $(1,0)$:

$$\frac{y-0}{x-1} = -1 \Rightarrow y = -1(x-1) \Rightarrow \boxed{y = -x+1}$$

(3) (a) $f'(x) = -\sin\left(\sqrt[3]{x+1}\cdot\frac{\pi}{2}\right)\cdot\frac{\pi}{2}\cdot\frac{1}{3}(x+1)^{-2/3}$ (chain rule)

$f'(0) = -\sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{2}\cdot\frac{1}{3}\cdot 1^{-2/3} = -\frac{\pi}{6}$

Linearization: $y = f(0) + f'(0)(x-0)$

$y = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{6}(x-0)$ ~~$\cos 1 - 1$~~ $= -\frac{\pi}{6}x$

Conclusion: linearization at $(0,-1)$ is $\boxed{y = -\frac{\pi}{6}x}$

(b) $\cos\left(\sqrt[3]{1.1}\,\frac{\pi}{2}\right) \sim y(0.1) = -\frac{\pi}{6}\cdot(0.1) = -\frac{0.1\pi}{6}$

(4) $f(x) = \frac{x^2+3}{x-1}$, $x \neq 1$

$$\begin{array}{r|l} & x+1 \\ x-1 & x^2+1 \\ & -x^2+x \\ \hline & x+1 \\ & -x+1 \\ \hline & 2 \end{array}$$

$f(x) = x+1+\frac{4}{x-1}$

(a) $f'(x) = 1 - \frac{4}{(x-1)^2}$

Critical points: $f'(x) = 0 \Rightarrow \frac{4}{(x-1)^2} = 1 \Rightarrow (x-1)^2 = 4$

$\Rightarrow x-1 = \pm 2 \Rightarrow x = -1$ and $x = 3$

x		-1		1		3	
y'	$+$	0	$-$	||	$-$	0	$+$
y	↗		↘	||	↘		↗

$x=-1$ local max
$y = f(-1) = \frac{4}{-2} = -2$

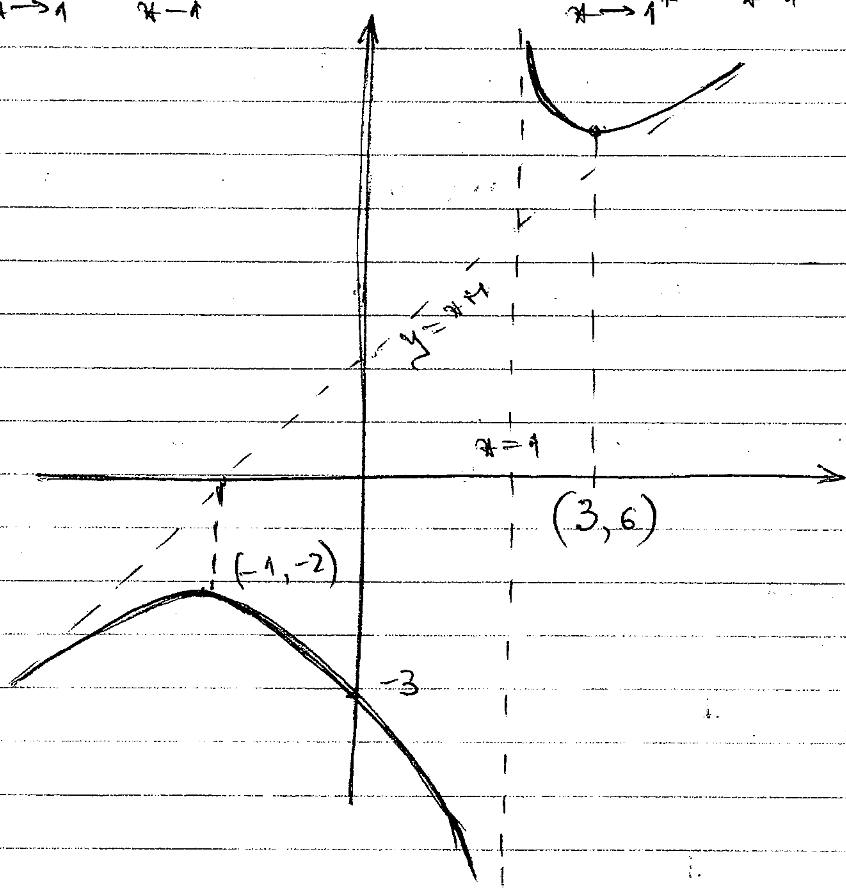
$x = 3$ local min.
$f(3) = \frac{12}{2} = 6$

Oblique asymptote : $y = x+1$ (dominant term as $x \to \pm\infty$)

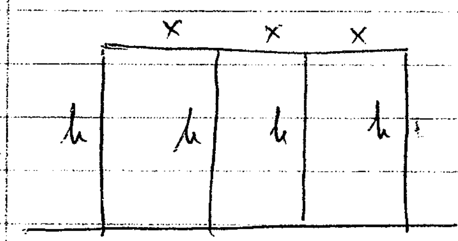
$\Rightarrow$ No horizontal asymptotes

Candidate for vertical asymptote : $x = 1$

$\lim_{x \to 1^-} \frac{x^2+3}{x-1} = -\infty$ and $\lim_{x \to 1^+} \frac{x^2+3}{x-1} = +\infty$ $\Rightarrow$ $x = 1$ V. A.



(5)



$4h + 3x = 10 \Rightarrow h = \frac{10-3x}{4}$

$A = xh$

$A(x) = x \cdot \frac{10-3x}{4} \Rightarrow A(x) = \frac{1}{4} x(10-3x) \rightarrow$ parabola with a max.

at the endpoint of $\left[0, \frac{10}{3}\right] \Rightarrow \boxed{x = \frac{5}{3}}$ yards $\boxed{h = \frac{5}{4} = 1.25}$ yards