APPM 1350 Spring 2009

Midterm III – Solutions

1. (a) $\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

Partition: $1, \frac{3}{2}, 2, \frac{5}{2}, 3.$

$$L_4 = \frac{1}{2}\left(f(1) + f\left(\tfrac{3}{2}\right) + f(2) + f\left(\tfrac{5}{2}\right)\right) = \frac{1}{2}\cdot\left(1+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}\right)$$

(b) $$T_4 = \frac{\frac{1}{2}}{2}\left(f(1) + 2f\left(\tfrac{3}{2}\right) + 2f(2) + 2f\left(\tfrac{5}{2}\right) + f(3)\right) =$$

$$= \frac{1}{4}\left(1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{1}{3}\right)$$

(c) L_4 overestimate, because f is decreasing $\left(f'(x) = -\frac{1}{x^2} < 0\right)$

T_4 overestimate, because f is concave up $\left(f''(x) = \frac{2}{x^3} > 0\right)$

(d) $f''(x) = \frac{2}{x^3}$ decreasing $\Rightarrow M = f''(1) = \frac{2}{1} = 2$

we need $\frac{b-a}{12}\cdot h^2 \cdot M \le 3\cdot 10^{-4} \Rightarrow$

$$\frac{3-1}{12}\cdot\left(\frac{3-1}{n}\right)^2\cdot 2 \le 3\cdot 10^{-4} \Rightarrow$$

$$\frac{1}{6}\cdot\frac{4}{n^2}\cdot 2 \le 3\cdot 10^{-4} \Rightarrow \frac{4\cdot\cancel{2}}{{}_3\cancel{6}\cdot 3\cdot 10^{-4}} \le n^2$$

$$\Rightarrow n^2 \ge \frac{4}{9\cdot 10^{-4}} = \frac{40000}{9} \Rightarrow n \ge \sqrt{\frac{40000}{9}} = \frac{200}{3} = 66.\overline{6}$$

$$\Rightarrow \boxed{n \ge 67}$$

(2) (a) $$\int \frac{\sqrt[3]{v}-1}{\sqrt{v}}\,dv = \int \left(v^{1/3}-1\right) v^{-1/2}\,dv = \int \left(v^{1/3-\frac{1}{2}} - v^{-1/2}\right) dv$$

$$= \int \left(v^{-1/6} - v^{-1/2}\right) dv = \frac{v^{5/6}}{5/6} - \frac{v^{1/2}}{1/2} + C = \frac{6}{5} v^{5/6} - 2 v^{1/2} + C$$

(b) $$\int_{\pi/3}^{\pi/2} \sin\theta \cos^3\theta \, d\theta = \int_{1/2}^{0} \cancel{\sin\theta} \cdot u^3 \cdot \frac{du}{-\cancel{\sin\theta}} = -\int_{1/2}^{0} u^3\,du = \int_{0}^{1/2} u^3\,du =$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta \implies d\theta = du/-\sin\theta$

$u\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$u\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$$= \left.\frac{u^4}{4}\right|_0^{1/2} = \frac{1}{2^4 \cdot 4} - 0 = \frac{1}{16 \cdot 4} = \boxed{\frac{1}{64}}$$

(c) $$\int_0^1 x^2 \sqrt{1-x^3}\,dx = \int_1^0 \cancel{x^2} \sqrt{u} \cdot \frac{du}{-3\cancel{x^2}} = \frac{1}{3} \int_0^1 u^{1/2}\,du = \frac{1}{3} \left.\frac{u^{3/2}}{3/2}\right|_0^1 =$$

$u = 1 - x^3$

$du = -3x^2\,dx \implies dx = \frac{du}{-3x^2}$

$u(0) = 1$

$u(1) = 0$

$$= \frac{1}{3} \cdot \frac{2}{3} \left. u^{3/2} \right|_0^1 = \frac{2}{9}(1-0) = \boxed{\frac{2}{9}}$$

(3) (a) $$\frac{dh}{dx} = \frac{d}{dx}\left[h(x)\right] = \frac{d}{dx}\left[\cos x + \int_0^{\sqrt{x}} t^3 \sin t^2\,dt\right] =$$

$$= -\sin x + \frac{d}{dx}\left[\int_0^{\sqrt{x}} t^3 \sin t^2\,dt\right] \xrightarrow[\text{part I}]{\text{FTC}} -\sin x + (\sqrt{x})^{\cancel{3}\,2} \sin(\sqrt{x})^2 \cdot \frac{1}{2\cancel{\sqrt{x}}}$$

$= -\sin x + \frac{1}{2} x \sin x = \sin x \left(\frac{1}{2}x - 1\right)$

(c) Critical points of f: $\sin x \left(\frac{1}{2}x - 1\right) = 0$ if either $x = 2$ or

$\sin x = 0 \iff x = 0, \pi.$

Critical points: $x = 0,\ x = 2,\ x = \pi.$

(4) $f(x) = \frac{1}{\pi} \cdot \sqrt{4 - x^2}$

(a) $y = \sqrt{4 - x^2}$, $-2 \le x \le 2$ $\rightsquigarrow$ the upper half of the circle centred at the origin and of radius $r = 2$ $\Rightarrow$ $\int_0^2 \sqrt{4 - x^2}\,dx$ = a fourth of the area ~~of~~ inside this circle $= \frac{1}{4} \cdot \pi \cdot r^2 = \frac{1}{\cancel{4}} \pi \cdot \cancel{2}^2 = \pi$

$$\int_0^2 \frac{1}{\pi} \sqrt{4 - x^2}\,dx = \frac{1}{\pi} \int_0^2 \sqrt{4 - x^2}\,dx = \frac{1}{\pi} \cdot \pi = \boxed{1} \Rightarrow f_{ave} = \frac{1}{2-0} \cdot 1 = \boxed{\frac{1}{2}}$$

(b) $f(c) = f_{ave}$

$\frac{1}{\pi}\sqrt{4 - c^2} = \frac{1}{2} \Rightarrow \sqrt{4 - c^2} = \frac{\pi}{2} \Rightarrow 4 - c^2 = \left(\frac{\pi}{2}\right)^2 \Rightarrow c^2 = 4 - \left(\frac{\pi}{2}\right)^2$

$c = \pm\sqrt{4 - \pi^2/4}$
But c in $[0, 2]$ $\Bigg\} \Rightarrow \boxed{c = \sqrt{4 - \frac{\pi^2}{4}}}$

(5) (a) $f(x) = \frac{x}{x-1}$, domain (f): $x \neq 1$ (i.e., $(-\infty, 1) \cup (1, \infty)$)

(b) Solve the equation $y = f(x)$ for x.

$y = \frac{x}{x-1} \Rightarrow y(x-1) = x \Rightarrow$

$yx - y = x \Rightarrow yx - x = y \Rightarrow x(y-1) = y.$

If $y = 1$, the equation becomes : $0 \cdot x = 1$, so it has no solutions $\Rightarrow$
$y = 1$ is not in the range of f.

If $y \neq 1 \Rightarrow x = \frac{y}{y-1}$ (the equation has a unique solution).

Conclusion : The function $f(x) = \frac{x}{x-1}$, with domain $x \neq 1$ and range $y \neq 1$ is one-to-one, and its inverse is $f^{-1}(x) = \frac{x}{x-1}$ (f is its own inverse!)

(c) domain f^{-1} = range $f = \{x \neq 1\}$
Range f^{-1} = domain $f = \{y \neq 1\}$.