1350 Exam 1

Answer Key

9-23-09

1.a) $\lim_{x\to 1}\left[\frac{1}{x-1} - \frac{1}{x^2-3x+2}\right] = \lim_{x\to 1}\left[\frac{1}{x-1} - \frac{1}{(x-1)(x-2)}\right] = \lim_{x\to 1}\left[\frac{x-2 \ -1}{(x-2)(x-1)}\right]$

$= \lim_{x\to 1}\left[\frac{x-3}{(x-2)(x-1)}\right]$ is $+$ or $-$ ∞.

Check L & R limits.

L: $\lim_{x\to 1^-} \frac{x-3}{(x-1)(x-2)} = -\infty$

R: $\lim_{x\to 1} \frac{x-3}{(x-1)(x-2)} = +\infty$

LH & RH Limits $\neq$ $\Rightarrow$ $\boxed{\text{DNE.}}$

1b) $\lim_{x\to 4^+} \frac{4-v}{|4-v|}$ Rewrite Piecewise $\frac{4-v}{|4-v|} = \begin{cases} \frac{4-v}{4-v} = 1 & v<4 \\ \frac{4-v}{-(4-v)} = -1 & v>4 \end{cases}$

Limit from the right is $v > 4$

so becomes

$\lim_{x\to 4^+} \frac{4-v}{|4-v|} = \lim_{x\to 4^+} \frac{4-v}{-(4-v)} = \boxed{-1}$

1c) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x^2\cos\left(\frac{1}{x^2}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$

Use Sandwich Theorem (i.e. Squeeze Thm)

we know

$-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$

so $-x^2 \leq x^2\cos\left(\frac{1}{x^2}\right) \leq x^2$

Since $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$

we $\boxed{\text{have } \lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right) = 0 \text{ By Sandwich Theorem}}$

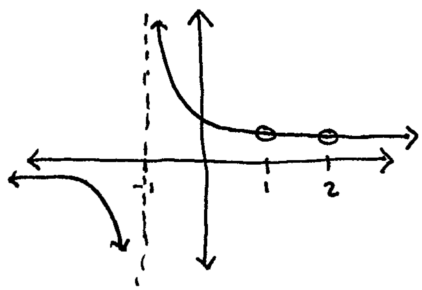
2\. a) Let $f(x) = \dfrac{x^2-3x+2}{(x^2-1)(x-2)}$. Factor: $f(x) = \dfrac{(x-1)(x-2)}{(x-1)(x+1)(x-2)}$

So $f(x) = \dfrac{1}{x+1}$ with $x \neq -1, 1, 2$

Domain: $\{x \in \mathbb{R} \mid x \neq 1, -1, 2\}$ set notation

or write D: $(-\infty, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$ interval notation

2b) Graph. Use simplified form $f(x) = \dfrac{1}{x+1}$



2c) $f(x)$ is continuous at a point c if
$f(c) = \lim\limits_{x \to c} f(x)$.

2d) $f(x)$ is discontinuous at $x = 1, -1$ and 2. All three fail the continuity test because $f(x)$ does not exist at these points.

2e) Same as part (a) because Rational functions are continuous on their Domain.
$\{x \in \mathbb{R} \mid x \neq 1, -1, 2\}$

or write $(-\infty, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$

3. $y = \sqrt{2x-1}$

Using the definition:

conjugate ↓

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \cdot \frac{\sqrt{2(x+h)-1}+\sqrt{2x-1}}{\sqrt{2(x+h)-1}+\sqrt{2x-1}}$$

$$= \lim_{h\to 0} \frac{(2x+2h-1)-(2x-1)}{h\left(\sqrt{2(x+h)-1}+\sqrt{2x-1}\right)} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left[\sqrt{2(x+h)-1}+\sqrt{2x-1}\right]}$$

evaluate the Limit $$= \frac{2}{\sqrt{2(x+0)-1}+\sqrt{2x-1}} = \frac{2}{2\sqrt{2x-1}} = \boxed{\frac{1}{\sqrt{2x-1}}}$$

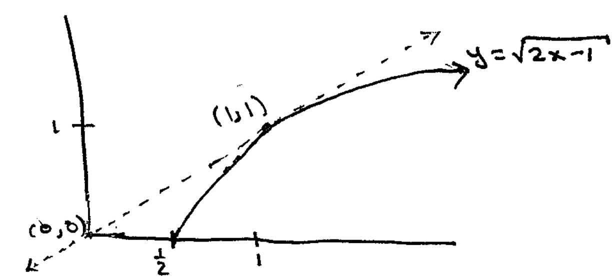
3b. At $x=5$ $\quad \left.\frac{dy}{dx}\right|_5 = \frac{1}{\sqrt{2(5)-1}} = \frac{1}{3}$ slope of tangent Line

at $x=5$ $\quad y = \sqrt{2(5)-1} = 3 \Rightarrow (5,3)$ point on curve

Tangent Line	$(y-3) = \frac{1}{3}(x-5)$
Normal Lime	$(y-3) = -3(x-5)$

3c) Diagram

Yes.



Line has slope 1

$\boxed{y = x \text{ is tangent to } y=\sqrt{2x-1} \text{ at } (1,1) \text{ and passes through } (0,0)}$

* See last page for Details

4 a) $y = x^3 + 3x^{-2} - 4\sqrt{x} + \pi$

$$\frac{dy}{dx} = 3x^2 - 6x^{-3} - 2x^{-1/2}$$

4 b) $g(x) = (x^3 - x^2 f(x))(2 - \frac{1}{x})$ $\qquad f(1)=2, \quad f'(1) = 5$

$$g'(x) = [3x^2 - (2x f(x) + x^2 f'(x))](2 - \tfrac{1}{x}) + (x^3 - x^2 f(x))(\tfrac{1}{x^2})$$

$$g'(1) = (3 - (2\cdot 2 + 1\cdot 5))(2-1) + (1 - 1\cdot 2)(1) = \boxed{-7}$$

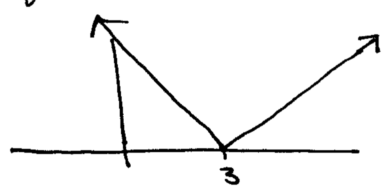
4c) $y = \dfrac{x^2+3}{5-2x^3} \Rightarrow \dfrac{dy}{dx} = \dfrac{2x(5-2x^3) - (-6x^2)(x^2+3)}{(5-2x^3)^2}$

simplifies to $\dfrac{dy}{dx} = \dfrac{2x^4+18x^2+10x}{(5-2x^3)^2}$

5a) False. Points $(1,0)$ and $(-1,2)$ $d = \sqrt{(-1-1)^2+(2-0)^2} = \sqrt{8} < 3.$
so $(1,0)$ is closer than 3 to center $(-1,2) \Rightarrow$ Inside the circle

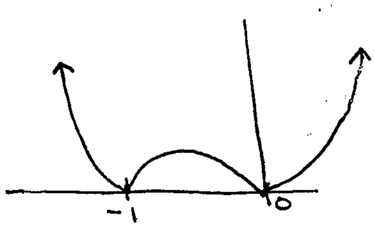
5b) False. The intermediate value theorem requires $f(x)$ to be CONTINUOUS.

5c) False. Counter example: $f(x) = |x-3|$
is continuous but not differentiable at $x = 3$.



5d) False. $\frac{d}{dx}(fg) = f'g + g'f$ Product Rule

5e) False. $f(x) = |x^2+x| = \begin{cases} x^2+x & \text{if } x^2+x \geq 0 \\ -(x^2+x) & \text{if } x^2+x < 0 \end{cases}$



$= \begin{cases} x^2+x & \text{if } x \leq -1 \text{ or } x \geq 0 \\ -(x^2+x) & \text{if } -1 < x < 0 \end{cases}$

Thus $f'(x) = \begin{cases} 2x+1 & \text{if } x < 1 \text{ or } x > 0 \\ -2x-1 & \text{if } -1 < x < 0 \end{cases}$

5f) True. if $f'(r)$ exists, then f is differentiable at r.
Differentiable functions are necessarily Continuous
so $\lim_{x \to r} f(x) = f(r)$, which is the definition of continuity.

3c. Derivation of Eqn. of Tangent Line

Any line tangent to the curve $y=\sqrt{2x-1}$ will pass through a point on the curve with coordinates

$$(x, \sqrt{2x-1})$$

and have slope $\dfrac{1}{\sqrt{2x-1}}$ as found in part (a).

If this tangent line passes through (0,0) as well we can solve the equation involving the two points for x

$$y_2 - y_1 = m(x_2 - x_1)$$

$$(\sqrt{2x-1} - 0) = \frac{1}{\sqrt{2x-1}}(x - 0)$$

then $2x-1 = x$

giving $x = 1$

Then $y = \sqrt{2(1)-1} = 1$

So the point on the curve is (1,1).

The resulting line then has slope $\dfrac{1}{\sqrt{2(1)-1}} = 1$

So the eqn. for the tangent line is

$$\boxed{y = x}$$ ■