APPM 1350 - Exam 2 Solutions - Fall 2009

1(a) False. Critical points of a fcn f occur when $f'(c)=0$ or when $f(c)$ exists but $f'(c)$ does not exist.

(b) False. $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, $f(0)=2$ and $f'(x)=3x^2-4 \Rightarrow f'(0)=-4$

$\therefore x_1 = 0 - \dfrac{2}{-4} = \dfrac{2}{4} = \dfrac{1}{2}$

(c) True. $h(x) = f(x)g(x)$

$h'(x) = f'(x)g(x) + f(x)g'(x)$. Since $f'(c)=0$ & $g'(c)=0 \Rightarrow h'(c)=0$

so $x=c$ is a critical point of h.

$h''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$

$h''(c) = f''(c)g(c) + f(c)g''(c)$.

Since $x=c$ is a min for f & g, $f''(c)>0$ & $g''(c)>0$

Since we are given $g(c)<0$ & $f(c)<0$

$\Rightarrow h''(c)<0 \Rightarrow x=c$ is a local max of $h(x)$.

2(a) $y = \sin(\sqrt{\theta})\tan(\theta^{-1}) \Rightarrow \dfrac{dy}{d\theta} = \dfrac{1}{2\sqrt{\theta}}\cos(\sqrt{\theta})\tan(\theta^{-1}) - \dfrac{1}{\theta^2}\sin(\sqrt{\theta})\sec^2(\theta^{-1})$

(b) $\displaystyle\lim_{x\to 0}\frac{\tan(3x)}{\sin(8x)} = \lim_{x\to 0}\frac{3x\,\frac{\tan(3x)}{3x}}{8x\,\frac{\sin(8x)}{8x}} = \boxed{\frac{3}{8}}$ since $\displaystyle\lim_{x\to 0}\frac{\sin(8x)}{8x}=1$

and $\displaystyle\lim_{x\to 0}\frac{\tan(3x)}{3x} = \lim_{x\to 0}\frac{\sin(3x)}{3x\cos(3x)} = 1$

(c) $x + \tan(xy) = 0$

Differentiate both sides with respect to x:

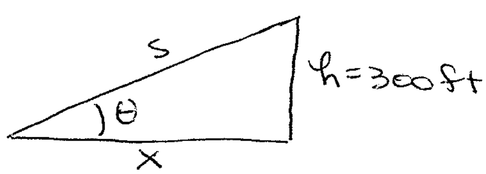
$1 + \sec^2(xy)\left(y + x\dfrac{dy}{dx}\right) = 0$

$\sec^2(xy)\left(y + x\dfrac{dy}{dx}\right) = -1$

$y + x\dfrac{dy}{dx} = -\cos^2(xy)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-\cos^2(xy) - y}{x}$

3.



Given: $\dfrac{dx}{dt} = 25\,\dfrac{ft}{sec}$ (and $\dfrac{dh}{dt}=0$ since h is not changing.)

Find: $\dfrac{d\theta}{dt}$ when $x = 400$ ft.

Note: $s = 500$ ft when $x = 400$ ft and $h = 300$ ft.

Equation: $\tan\theta = \dfrac{h}{x}$.

Differentiate both sides with respect to t:

$\sec^2\theta\,\dfrac{d\theta}{dt} = h(-x^{-2})\dfrac{dx}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{-h}{x^2}\cos^2\theta\,\dfrac{dx}{dt}$

$\left.\dfrac{d\theta}{dt}\right|_{x=400} = -\dfrac{300}{500^2}\cdot\underbrace{\left(\dfrac{400}{500}\right)^2}_{\cos^2\theta}\cdot 25\,\dfrac{ft}{sec}$

$= \dfrac{-3}{100}\,\dfrac{\text{radians}}{\text{sec}}$

4. $f'(x) = (x-2)^2(x+3)^2$

(a) $f'(x) \geq 0$ for all x. $\therefore f(x)$ is increasing on $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
or $(-\infty, \infty)$ was also acceptable

(b) Since $f(x)$ is increasing for all x & is continuous everywhere, $f(x)$ has no max or min. (Note: $x = -3$ & $x = 2$ are the critical points for $y = f(x)$ but they are neither max nor mins.)

(c) $f''(x) = 2(x-2)(x+3)^2 + 2(x-2)^2(x+3) = 2(x-2)(x+3)[x+3+x-2]$
$\uparrow$ factor!
$= 2(x-2)(x+3)(2x+1)$

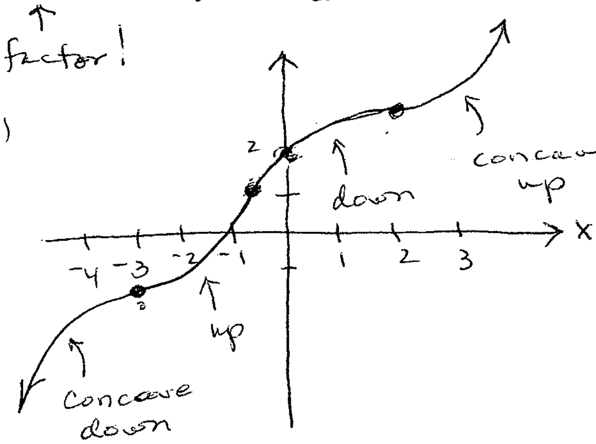
$f''(x) = 0$ when $x = -3, -1/2, 2$

$f''(x)$: $-$ | -3 | $+$ | $-1/2$ | $-$ | 2 | $+$
$\cap$ -3 $\cup$ $-1/2$ $\cap$ 2 $\cup$

Concave down: $(-\infty, -3) \cup (-\frac{1}{2}, 2)$

(d) $\underline{\underline{3}}$ inflection pts: $x = -3, -\frac{1}{2}, 2$

(e)



5. $f(x) = \dfrac{x^2 - 49}{x^2 + 5x - 14} = \dfrac{(x-7)(x+7)}{(x-2)(x+7)}$

horizontal asymptote: $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{(x^2 - 49)(\frac{1}{x^2})}{(x^2 + 5x - 14)(\frac{1}{x^2})} = \lim\limits_{x \to \infty} \dfrac{1 - \frac{49}{x^2}}{1 + \frac{5}{x} - \frac{14}{x^2}} = 1$

similarly: $\lim\limits_{x \to -\infty} f(x) = 1$

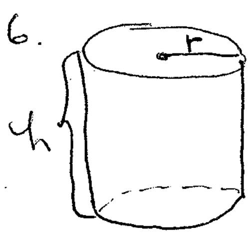
$\therefore y = 1$ is a horizontal asymptote

vertical asymptote: $\lim\limits_{x \to 2^+} \dfrac{x-7}{x-2} = -\infty$ and $\lim\limits_{x \to 2^-} \dfrac{x-7}{x-2} = +\infty$

$\therefore x = 2$ is a vertical asymptote.

(Note: $x = -7$ is a removable discontinuity, not a vertical asymptote.)

6.



Given: $V = \pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2}$

Want: Minimize cost.

Cost = (cost of top + bottom) + cost of cardboard side
$= 3 \cdot 2\pi r^2 + 1 \cdot 2\pi r h = 6\pi r^2 + 2\pi r h$

3 cents

Substitute: $h = \dfrac{1000}{\pi r^2}$

$\Rightarrow C(r) = 6\pi r^2 + 2\pi r\left(\dfrac{1000}{\pi r^2}\right) = 6\pi r^2 + \dfrac{2000}{r}$

$C'(r) = 12\pi r - \dfrac{2000}{r^2}$

$C'(r) = 0$ when $\dfrac{2000}{r^2} = 12\pi r$

$\Rightarrow r = \left(\dfrac{500}{3\pi}\right)^{1/3}$

$C''(r) = 12\pi + \dfrac{4000}{r^3} > 0$ for all $r > 0$

$\therefore r = \left(\dfrac{500}{3\pi}\right)^{1/3}$ is a min

$h = \dfrac{1000}{\pi\left(\frac{500}{3\pi}\right)^{2/3}}$