1350 Midterm #1 Spring 2010

Solution Key

1. a) $g(t) = \dfrac{t^2-4}{t^3+7t-1}$ use quotient rule

$$\frac{dg}{dt} = \frac{(2t-4)(t^3+7t-1) - (3t^2+7t-1)(t^2-4)}{(t^3+7t-1)^2}$$

b) $f(x) = (x + x^{-2/3})(x - x^{-2} + 1)$

$$\frac{df}{dx} = \left(1 - \tfrac{2}{3}x^{-5/3}\right)\left(x - x^{-2} + 1\right) + \left(1 + 2x^{-3}\right)\left(x + x^{-2/3}\right)$$

2. a) $\lim_{x \to 2^-} f(x) = 1$ (No justification necessary)

b) $\lim_{x \to 1^+} f(x) = 0$ (No justification necessary)

c) $\lim_{x \to 1} f(x) = \text{DNE}$ L & R limits do not agree.

d) No. For example, $\lim_{x \to 1} f(x)$ DNE.

e) $f(x)$ is discontinuous at $x=1$ and $x=2$.

* at $x=1$ $\lim_{x \to 1} f(x)$ DNE

* at $x=2$ $\lim_{x \to 2} f(x) \neq f(2)$

3. a) For a function $f(x)$ the derivative is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ if this limit exists.

b) $f(x) = \frac{4}{x^2}$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{\frac{4}{(x+h)^2} - \frac{4}{x^2}}{h} = 4 \cdot \lim_{h \to 0} \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h}$$

$$= 4 \cdot \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{h \cdot x^2 (x+h)^2} = 4 \cdot \lim_{h \to 0} \frac{-\cancel{h}(2x - h)}{\cancel{h} \cdot x^2 (x+h)^2}$$

$$= \frac{4 \cdot (-2x)}{x^4} = \boxed{\frac{-8}{x^3}}$$

c) at $(x_0, y_0) = (2, 1)$ we have $\left.\frac{df}{dx}\right|_{x=2} = \frac{-8}{2^3} = -1$

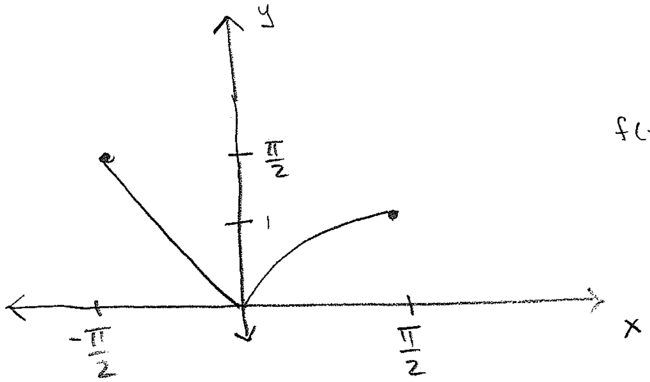
so line has equation

$\boxed{y - 1 = -1(x - 2)}$ ← point-slope form

OR

$\boxed{y = -x + 3}$ ← slope-intercept form

4. a)



$$f(x) = \begin{cases} \sin x & 0 < x \leq \frac{\pi}{2} \\ -x & -\frac{\pi}{2} \leq x \leq 0 \end{cases}$$

b) Yes. $f(x)$ is continuous at $x=0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \sin x = 0$$

$$\therefore \lim_{x\to 0} f(x) = 0$$

Also $f(0) = -0 = 0$

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

$\therefore$ $f(x)$ continuous at $x=0$.

4 c) No. $f(x)$ is not differentiable at $x=0$.

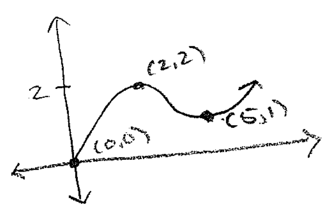
$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-(x+h) - (-x)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{\sin(x+h) - \sin x}{h} = 1 \text{ @ } x=0$$

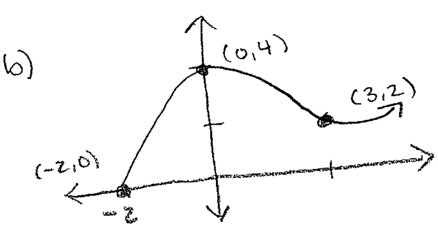
$$\therefore \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \neq \lim_{x\to 0^+} \frac{f(x+h)-f(x)}{h}$$

$\therefore$ Not Differentiable at $x=0$.

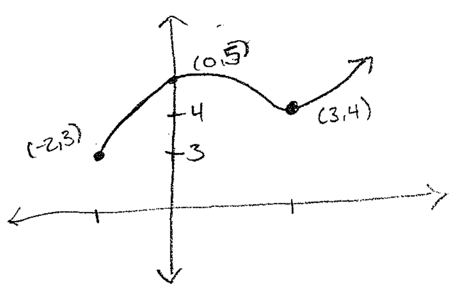
5. a)



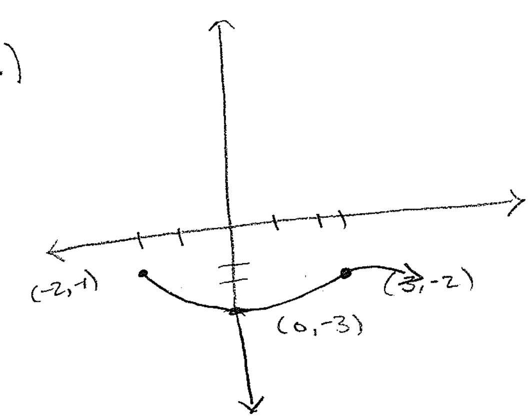
b)



c)



d)



6. a) False. Continuity is not guaranteed at 0 if f is not continuous at $g(0)$

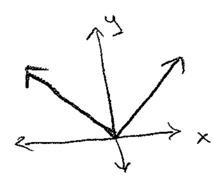
Ex: $f(x) = \frac{1}{x+1}$, $g(x) = x - 1$

f, g both continuous at $x=0$ but

$f(g(x)) = \frac{1}{(x-1)+1} = \frac{1}{x}$ NOT continuous at $x=0$.

b) False. Continuity is necessary but not sufficient for differentiability.

Ex: $f(x) = |x|$



is continuous at $x=0$ but not Differentiable at $x=0$.