

1. (21 pts) Each part is worth 7 points.

(a) Find the derivative of $g(x) = \cot(x^2) + \cot^2(x)$

(b) Find an equation of the tangent line to the curve $y = \frac{|x|}{\sqrt{2-x^2}}$ when $x = 1$.

(c) Find an equation of the tangent line to the curve $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.

Solution:

(a) $f'(x) = -\csc^2(x^2) \cdot 2x + 2\cot(x) \cdot -\csc^2(x) = -2x\csc^2(x^2) - 2\cot(x)\csc^2(x)$

(b) Note that if $x > 0$ then $y = \frac{x}{\sqrt{2-x^2}}$ and $y' = \frac{2}{(2-x^2)^{3/2}}$, so $y'(1) = 2$ and $y(1) = 1$, so the equation of the tangent line is $y = 1 + 2(x - 1) = -1 + 2x$

(c) Using implicit differentiation we have $2x + y + xy' + 2yy' = 0$, so $y' = \frac{-(2x + y)}{x + 2y}$ and $y'(1, 1) = -1$ and so the equation of the tangent line is $y = 1 + (-1)(x - 1) = 2 - x$.

2. (a) (10 pts) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ at $x = 3$.

(b) (10 pts) The radius of a circular disk is given as 4 cm with an error in measurement of ± 0.2 cm. Use differentials to determine the percentage error in the calculated area of the disk.

Solution:

(a) Note that $f'(x) = \frac{-x}{\sqrt{25 - x^2}}$ and $L(x) = f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3) = \frac{25}{4} - \frac{3}{4}x$

(b) Note that $A = \pi r^2$ and so $dA = 2\pi r dr$ and thus

$$\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2dr}{r} = \frac{2(\pm 0.2)}{4} = \pm 0.1$$

so the relative error is ± 0.1 and the percentage error is $\pm 10\%$.

3. (10 pts) Find the absolute maximum and absolute minimum values of $f(x) = x\sqrt{4 - x^2}$ on $[-1, 2]$

Solution:

Here $f'(x) = \frac{4 - 2x^2}{\sqrt{4 - x^2}}$ so $f'(x) = 0$ when $x = \pm\sqrt{2}$ and $f'(x)$ is undefined when $x = \pm 2$ so the critical points are $x = \pm\sqrt{2}, \pm 2$.

The critical points in our interval are $x = \sqrt{2}, 2$, and $f(-1) = -\sqrt{3}$, $f(2) = 0$ and $f(\sqrt{2}) = 2$.

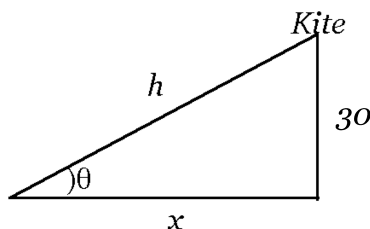
So the absolute minimum value is $-\sqrt{3}$ and the absolute maximum value is 2.

4. (10 pts) A kite 30 ft above the ground moves horizontally at a speed of 10 ft/s. At what rate is the angle between the kite-string and the ground decreasing when 50 ft of string has been let out?

Solution:

Let x be the horizontal distance the kite has traveled and let h be the length of string that has been let out. Then we know $dx/dt = 10$ ft/s. Now note that if θ is the angle that the string makes with the ground then we have $\cot(\theta) = x/30$ and so

$$\frac{d\theta}{dt} = -\sin^2(\theta) \cdot \frac{dx}{dt} \cdot \frac{1}{30} = -\left(\frac{30}{50}\right)^2 \cdot 10 \cdot \frac{1}{30} = -\frac{3}{25} \text{ rad/s}$$



5. (21 pts) Each part is worth 7 points.

- (a) Find constants F and M such that the function $y = F \sin(x) + M \cos(x)$ satisfies the differential equation $y'' + y' = \sin(x)$.
- (b) Suppose $f(x)$ is differentiable for all values of x . Let $B(x) = f(x) \circ \cos(x)$, find an expression for $B''(x)$.
- (c) Use the Mean Value Theorem on the function $f(x) = \sqrt{1+x}$, for $0 \leq x \leq 1$, to show that $\sqrt{2} < 3/2$.

Solution:

- (a) Note that $y' = F \cos(x) - M \sin(x)$ and $y'' = -F \sin(x) - M \cos(x)$ and so,

$$y'' + y' = (-F - M) \sin(x) + (F - M) \cos(x) = \sin(x)$$

so $-F - M = 1$ and $F - M = 0$, and so $F = M = -\frac{1}{2}$.

- (b) Here $B(x) = f(\cos(x))$ and so, $B'(x) = f'(\cos(x)) \cdot -\sin(x)$, and

$$B''(x) = f''(\cos(x)) \sin^2(x) - f'(\cos(x)) \cos(x)$$

- (c) Note that $f'(x) = \frac{1}{2\sqrt{1+x}}$ and so by the Mean Value Theorem, there exists a c in $(0, 1)$ such that

$$\frac{f(1) - f(0)}{1 - 0} = f'(c), \text{ that is, } \sqrt{2} - 1 = \frac{1}{2\sqrt{1+c}} \text{ for some } c \text{ in } (0, 1)$$

and so $\sqrt{2} = 1 + \frac{1}{2\sqrt{1+c}}$ and note that $\frac{1}{2\sqrt{1+c}} < \frac{1}{2}$ since $\sqrt{1+c} > 1$ and so,

$$\sqrt{2} = 1 + \frac{1}{2\sqrt{1+c}} < 1 + \frac{1}{2} = \frac{3}{2}$$

and so we have $\sqrt{2} < 3/2$.

6. (18 pts) Given $f(x) = \frac{1 + 2x^2}{x^2 - 1}$, $f'(x) = \frac{-6x}{(x^2 - 1)^2}$, and $f''(x) = \frac{6(3x^2 + 1)}{(x^2 - 1)^3}$

- (a) Find the intervals on which $f(x)$ is increasing or decreasing and find any local maximum and minimum points of $f(x)$. (Give the x and y coordinate of any local max or min's.) Justify your answers.
- (b) Describe the concavity of $f(x)$, are there any inflection points? If so, what is the coordinate of the inflection point? (Give the x and y coordinate of any inflection points.) Justify your answers.
- (c) In your blue book, clearly sketch the graph of $f(x)$. Be sure to LABEL all horizontal and vertical asymptotes, intercepts, local maximums and minimums.

Solution:

- (a) Increasing on $(-\infty, -1) \cup (-1, 0)$; decreasing on $(0, 1) \cup (1, \infty)$; local maximum at $(0, -1)$
- (b) Concave up on $(-\infty, -1) \cup (1, \infty)$; concave down on $(-1, 1)$; there are no inflection points
- (c) The graph:

