APPM 1350 Midterm #3 Summer 2011

CAREFULLY PRINT, on the front of your bluebook: a grading key, your name, student ID, and instructor's name (Guinn) . This exam is worth 100 points and has 5 questions. **Show all work!** Answers with no justification will receive no points. Please begin each problem on a new page. No notes, calculators, or electronic devices are permitted.

1. (30 points) Evaluate the following expressions:

   (a) $\int_0^3 f(3x)\,dx$ if $\int_0^9 f(x)\,dx = -2$

   (b) $\int_0^{\sqrt{\pi/2}} x\sin(x^2)\sqrt{\cos(x^2)}\,dx$

   (c) $\int_{-2}^3 f(x)\,dx$ where $f(x) = \begin{cases} \sqrt{4-x^2} & -2 \le x \le 0 \\ 2 & 0 \le x \le 1 \\ 2 - x + x^2 & 1 \le x \le 3 \end{cases}$

   (d) $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{n+1}{n}\right)^4 + \left(\frac{n+2}{n}\right)^4 + \cdots + \left(\frac{n+n}{n}\right)^4\right]$ (Hint: Think definite integral.)

   (e) $\int x\sqrt{4-3x^2},\ dx$

2. (15 points)

   (a) State the Fundamental Theorem of Calculus.

   (b) Find the following derivative: $\frac{d}{dx}\left(\int_1^{x^2} t^3 - \frac{2}{t}\,dt\right)$

3. (20 points) Consider $\int_0^2 x^3 - x\,dx$.

   (a) Write down and evaluate a Riemann sum approximation to this integral by dividing $[0,2]$ into four equal subintervals and using the right endpoints of each subinterval for evaluation.

   (b) Now write down a Riemann sum to approximate the integral by dividing $[0,2]$ into n subintervals.

   (c) Evaluate the limit as $n \to \infty$ of your sum in part (b) (you can evaluate the integral to check your answer, but for full credit, you must take the limit of the sum).

   (d) Is your Riemann sum in part (a) an underestimate or an over estimate?

4. (20 points) Let $y(x) = \int_0^x \frac{1}{3+t+t^2}\,dt$

   (a) Find the interval(s) on which the curve is concave upward.

   (b) Determine the linearization, $L(x)$, of $y(x)$ near $x = 0$.

   (c) Apply one iteration of Newton's Method to find an approximation of a root of $y(x)$. Use $x_1 = 0$. Does your answer make sense?

5. (15 points) True/False. For this problem only, just write TRUE or FALSE, no explanation is required.

   (a) $\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$

   (b) $\int_{-1}^2 \frac{1}{x^2}\,dx = -\frac{3}{2}$

   (c) $\int x\sin x\,dx = \frac{-x^2}{2}\cos x + C$

   (d) If $f(x) \ge g(x)$ for all $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

   (e) If $f(x) \ge g(x)$ for all $a \le x \le b$, then $f'(x) \ge g'(x)$ for all $a \le x \le b$.

Solutions

1. (a) $\int_0^3 f(3x)\,dx$ if $\underbrace{\int_0^9 f(x)\,dx = -2}$

$u = 3x \qquad u(0) = 0$

$du = 3dx \qquad u(3) = 9$

$$\int_0^9 f(u)\,\frac{du}{3} = \frac{1}{3}\int_0^9 f(u)\,du = \frac{1}{3}(-2) = \boxed{\frac{-2}{3}}$$

(b) $\int_0^{\sqrt{\pi/2}} x \sin(x^2) \sqrt{\cos(x^2)}\,dx$

$u = \cos(x^2) \qquad u(0) = 1$

$du = -\sin(x^2)\, 2x\, dx \qquad u(\sqrt{\pi/2}) = 0$

$$\int_1^0 \sqrt{u}\,\frac{du}{-2} = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_0^1 = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$$

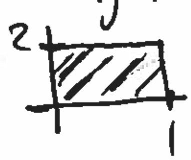
(c) $\int_{-2}^3 f(x)\,dx$, $f(x) = \begin{cases} \sqrt{4-x^2} & -2 \le x \le 0 \\ 2 & 0 \le x \le 1 \\ 2 - x + x^2 & 1 \le x \le 3 \end{cases}$

$$\underline{\int_{-2}^0 \sqrt{4-x^2}\,dx} + \underline{\int_0^1 2\,dx} + \underline{\int_1^3 2 - x + x^2\,dx}$$

area of a quarter of a circle w/ radius 2:

$\frac{\pi(2^2)}{4} = \pi$

area of a rectangle w/ base = 1, height = 2



$2 \times 1 = 2$

$$2x - \frac{x^2}{2} + \frac{x^3}{3}\Big|_1^3$$

$$= \left[2(3) - \frac{3^2}{2} + \frac{3^3}{3}\right] - \left[2 - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \left(15 - \frac{9}{2}\right) - \left(\frac{3}{2} + \frac{1}{3}\right)$$

$$= \frac{21}{2} - \frac{11}{6} = \frac{52}{6} = \frac{26}{3}$$

$$\boxed{\pi + 2 + \frac{26}{3}}$$

1. (d) $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{n+1}{n}\right)^4 + \left(\frac{n+2}{n}\right)^4 + \cdots + \left(\frac{n+n}{n}\right)^4\right]$

This is the definition of the definite integral

$$\int_1^2 x^4\,dx = \left.\frac{x^5}{5}\right|_1^2 = \frac{2^5}{5} - \frac{1^5}{5}$$

$$= \frac{32}{5} - \frac{1}{5} = \boxed{\frac{31}{5}}$$

Notice that $\Delta x = \frac{b-a}{n} = \frac{1}{n} \Rightarrow b-a = 1$

$f(x_i) = x_i^4$

$x_i = a + i\Delta x$

$a = 1,\ b = 2$

$x_1 = \frac{n+1}{n} = \frac{n}{n} + \frac{1}{n} = 1 + \frac{1}{n}$
$= 1 + \Delta x$

$x_2 = \frac{n+2}{n} = \frac{n}{n} + \frac{2}{n} = 1 + 2\Delta x$

$\vdots$

$x_n = \frac{n+n}{n} = \frac{n}{n} + \frac{n}{n} = 1 + n\Delta x = 2$

So we have

$$\lim_{n\to\infty} \Delta x\left[(x_1)^4 + (x_2)^4 + \cdots + (x_n)^4\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_1^2 x^4\,dx = \boxed{\frac{31}{5}}$$

for $a=1,\ b=2$

(e) $\int x\sqrt{4-3x^2}\,dx$

$u = 4-3x^2$

$du = -6x\,dx$

$$\int u^{1/2}\,\frac{du}{-6} = -\frac{1}{6}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$= \boxed{-\frac{u^{3/2}}{9} + C}$$

2. (a) If $f(x)$ is continuous on $[a,b]$ and F is an antiderivative of f $(F' = f)$ then:

I. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

II. $\int_a^b f(t)\,dt = F(b) - F(a)$

(b) $$\frac{d}{dx}\int_1^{x^2}\left(t^3 - \frac{2}{t}\right)dt = \left[(x^2)^3 - \frac{2}{x^2}\right]2x$$

$$= \left(x^6 - \frac{2}{x^2}\right)2x$$

$$= \boxed{2x^7 - \frac{4}{x}}$$

Note: $u = x^2$ $\quad \frac{du}{dx} = 2x$

$$\frac{d}{dx} I = \frac{dI}{du}\frac{du}{dx}$$

$$= \left(\frac{d}{du}\int_1^u t^3 - \frac{2}{t}\,dt\right)\frac{du}{dx} = \left(u^3 - \frac{2}{u}\right)2x$$

$$= \left((x^2)^3 - \frac{2}{x^2}\right)2x = 2x^7 - \frac{4}{x}$$

3. $\int_0^2 x^3 - x\,dx$

(a) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$ $\qquad x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$

$$I \approx \frac{1}{2}\left[f(x_1) + \cdots + f(x_4)\right]$$

$$= \boxed{\frac{1}{2}\left[\left(\frac{1}{8} - \frac{1}{2}\right) + (1-1) + \left(\frac{27}{8} - \frac{3}{2}\right) + (8-2)\right]}$$

$$= \frac{1}{2}\left[\frac{28}{8} - 2 + 6\right] = \left[\frac{14}{8} + 2\right] = \frac{7}{4} + \frac{8}{4} = \boxed{\frac{\text{[illegible]}}{4}}$$

3. (b) $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_1 = 0 + \Delta x = \frac{2}{n}$

$x_2 = 0 + 2\Delta x = \frac{4}{n}$

$\vdots$

$x_n = 0 + n\Delta x = \frac{2n}{n} = 2$

note that $x_i = \frac{2i}{n}$

$$\boxed{I \approx \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 - \frac{2i}{n}\right]}$$

(comes from $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

(c) $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 - \frac{2i}{n}\right] = \lim_{n\to\infty}\left[\frac{16}{n^4}\sum_{i=1}^{n} i^3 - \frac{4}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right]\right]$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{16}}{n^{\cancel{4}2}}\,\frac{\cancel{n^2}(n+1)^2}{4} - \frac{\cancel{4}}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)}{2}\right)$$

Use dominance of powers (equal powers of n in numerator and denominator)

$$= 4 - \cancel{4} = \boxed{\cancel{2}}$$

(d) Notice that in part (c) we evaluated $\int_0^2 x^3 - x\,dx$ directly. $\frac{15}{4} > 2$, so our estimate in (a) was an overestimate.

Note: $$\int_0^2 x^3 - x\,dx = \frac{x^4}{4} - \frac{x^2}{2}\Big|_0^2 = \frac{16}{4} - \frac{4}{2} = 4 - 2 = 2$$

is a check of our work in (c).

4. $y(x) = \int_0^x \frac{1}{3+t+t^2}\,dt$

(a) $y'(x) = \frac{1}{3+x+x^2}$ by the FTC

$$y''(x) = -\frac{1}{(3+x+x^2)^2}(2x+1)$$

$y(x)$ is concave up when $y''(x) > 0$

$y''(x) > 0$ when $-(2x+1) > 0$

$2x+1 < 0$

$\boxed{x < -\frac{1}{2}}$ or $\boxed{(-\infty, -\frac{1}{2})}$

(b) $L(x) = f(a) + f'(a)(x-a)$ here $a=0$

$$L(x) = \underbrace{\int_0^0 \frac{1}{3+t+t^2}\,dt}_{=0} + \frac{1}{3+0+0^2}(x-0)$$

$= 0$ because our limits of integration are the same

$\Rightarrow \boxed{L(x) = \frac{1}{3}x}$

(c) Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_2 = 0 - \frac{y(0)}{y'(0)} = 0 - \frac{0}{\frac{1}{3}} = \boxed{0 = x_2}$$

This makes sense because $y(0) = 0$, so we've already found the root and Newton's Method will just keep giving it back to us.

5. (a) $\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$ $\boxed{\text{FALSE}}$

(b) $\int_{-1}^{2} \frac{1}{x^2}\,dx = -\frac{3}{2}$ $\boxed{\text{FALSE}}$: Note that $\frac{1}{x^2}$ is not defined at $x=0$ which is part of our interval

(c) $\int x \sin x\,dx = \frac{-x^2}{2}\cos x + C$ $\boxed{\text{FALSE}}$

$$\frac{d}{dx}\left[-\frac{x^2}{2}\cos x + C\right] = -x\cos x + \frac{x^2}{2}\sin x$$
$$\neq x \sin x$$

(d) If $f(x) \geq g(x)$ for $a \leq x \leq b$

then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ $\boxed{\text{TRUE}}$

Interpreting the integrals as area under the curves gives a good visualization as to why this is true.

(e) If $f(x) \geq g(x)$ for $a \leq x \leq b$

then $f'(x) \geq g'(x)$ for $a \leq x \leq b$ $\boxed{\text{FALSE}}$

Counter example: $f(x) = 5$, $g(x) = x$ on $[0,4]$

$f(x) \geq g(x)$ on $[0,4]$ but $f'(x) = 0 < 1 = g'(x)$.