Solutions

1. (a) $\lim_{x\to\infty} x^{\ln 1/9}(1+\ln x)$ $\qquad x^{\ln 1/9} = x^{\ln(9)^{-1}} = x^{-\ln 9}$

Indeterminate: $0\cdot\infty$

$$= \lim_{x\to\infty} \frac{1+\ln x}{x^{\ln 9}} \quad \left(\frac{\infty}{\infty}\right)$$

(Note $\ln 9 \approx 3$
$(e^3 \approx 9)$
$\Rightarrow \ln 9 - 1 > 0)$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\ln 9\, x^{\ln 9 - 1}} = \frac{0}{\infty} = \boxed{0}$$

(b) $\lim_{x\to\infty} \left(1+\frac{r}{n}\right)^{nt}$ $\qquad$ Indeterminate: 1^∞

$$y = \left(1+\frac{r}{n}\right)^{ny} \Rightarrow \ln y = nt\ln\left(1+\frac{r}{n}\right)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1+\frac{r}{n}\right) \quad (\infty\cdot 0)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{r}{n}\right)}{\frac{1}{nt}} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{r}{n}}\, r\cancel{\left(-\frac{1}{n^2}\right)}}{\frac{1}{t}\cancel{\left(-\frac{1}{n^2}\right)}} = \lim_{n\to\infty} \frac{rt}{1+\frac{r}{n}} = rt$$

$$\lim_{n\to\infty} \ln y = rt \Rightarrow \lim_{n\to\infty} y = \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt} = \boxed{e^{rt}}$$

(c) $\lim_{x\to 0} \ln(\tan^{-1}(x))$ $\qquad \tan^{-1}(0) = \;\Rightarrow\; \tan\theta = 0$
$\theta = 0$

$$= \lim_{\tan^{-1}x \to 0} \ln(\tan^{-1}(x)) = -\infty \qquad \lim_{x\to 0} \ln x = -\infty$$

The limit does not exist and approaches $-\infty$

1. (d) $\lim_{h\to 0} \frac{1}{h} \int_1^{1+h} \sqrt{1+t^2}\,dt$ Indeterminate $(\infty \cdot 0)$

$$= \lim_{h\to 0} \frac{\int_1^{1+h} \sqrt{1+t^2}\,dt}{h} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{h\to 0} \frac{\sqrt{1+(1+h)^2}}{1} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2}}$$

2. (a) $y = 2^{\cosh x}(\ln x^2)$

$$\boxed{y' = 2^{\cosh x}\ln 2(\sinh x)(\ln x^2) + 2^{\cosh x}\left(\frac{1}{x^2}\right)(2x)}$$

(b) $x^2 y = e^{xy} - \log_3(\cos x)$

$$2xy + x^2 y' = e^{xy}(1y + xy') - \frac{1}{\cos x \ln 3}(-\sin x)$$

$$x^2 y' - e^{xy} x y' = e^{xy} y - 2xy + \frac{\tan x}{\ln 3}$$

$$y'(x^2 - xe^{xy}) = ye^{xy} + \frac{\tan x}{\ln 3} - 2xy$$

$$\boxed{y' = \frac{ye^{xy} + \frac{\tan x}{\ln 3} - 2xy}{x^2 - xe^{xy}}}$$

(c) $y = \frac{x(x^2-2)^{3/2}}{\sqrt{x+1}\,(x^3+4)} \Rightarrow \ln y = \ln x + \frac{3}{2}\ln(x^2-2) - \frac{1}{2}\ln(x+1) - \ln(x^3+4)$

$$\Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x^2-2}\right)(2x) - \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x^3+4}(3x^2)$$

$$\boxed{y' = \left(\frac{1}{x} + \frac{3x}{(x^2-2)} - \frac{1}{2(x+1)} - \frac{3x^2}{x^3+4}\right)\left(\frac{x(x^2-2)^{3/2}}{\sqrt{x+1}\,(x^3+4)}\right)}$$

3. (a) $\int x\, 3^{x^2}\, dx$ $\qquad u = x^2$, $du = 2x\,dx$

$$= \frac{1}{2}\int 3^u\, du = \frac{1}{2\ln 3} 3^u + C = \boxed{\frac{1}{2\ln 3} 3^{x^2} + C}$$

(b) $f(x) = \frac{\log_2 x}{x}$ on $[1,4]$ $\quad (\log_2 x = y \leftrightarrow x = 2^y)$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 \frac{\log_2 x}{x}\,dx$$

$u = \log_2 x \quad u(4) = 2$

$du = \frac{1}{x\ln 2}dx \quad u(1) = 0$

$$f_{ave} = \frac{1}{3}\int_0^2 u\,(\ln 2\, du) = \frac{\ln 2}{3}\int_0^2 u\,du = \frac{\ln 2}{3}\left.\frac{u^2}{2}\right|_0^2$$

$$= \frac{\ln 2}{3}\left[\frac{2^2}{2} - 0\right]$$

$$\boxed{f_{ave} = \frac{2\ln 2}{3}}$$

(c) $\int_1^x \frac{1}{3+t+t^2}\,dt$ near $x = 1$:

$L(x) = f(a) + f'(a)(x-a)$

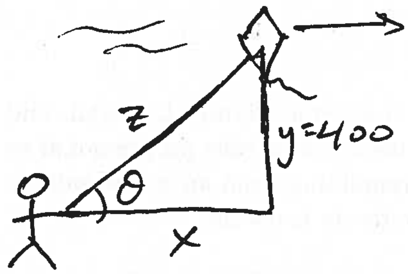
$f(1) = \int_1^1 \frac{1}{3+t+t^2}\,dt = 0$

$f'(x) = \frac{1}{3+x+x^2}$

$f'(1) = \frac{1}{3+1+1} = \frac{1}{5}$

$$\boxed{L(x) = \frac{1}{5}(x-1) = \frac{1}{5}x - \frac{1}{5}}$$

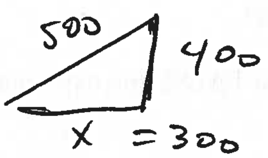
4. $20 \text{ ft/s} = \frac{dx}{dt}$



$$\theta = \tan^{-1}\frac{400}{x}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{400}{x}\right)^2}\left(-\frac{400}{x^2}\right)\frac{dx}{dt}$$

when $z = 500$,



$$\left.\frac{d\theta}{dt}\right|_{z=500} = \frac{1}{1+\left(\frac{400}{300}\right)^2}\left(-\frac{400}{(300)^2}\right)(20)$$

$$= \frac{1}{1+\frac{16}{9}}\left(-\frac{4}{900}\right)(20)$$

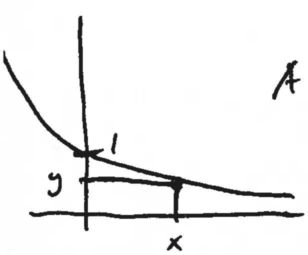
$$= \frac{1}{\frac{25}{9}}\left(-\frac{4}{900}\right)(20)$$

$$= \frac{9}{25}\left(-\frac{4}{900}\right)(20)$$

$$= -\frac{9(4)(4)}{5(900)}$$

$$= \frac{-16}{500} = \boxed{-\frac{4}{125}\ \frac{\text{rad}}{\text{s}}}$$

5. $y = e^{-x}$



$$A = xy = xe^{-x}$$

$$\frac{dA}{dx} = (1)e^{-x} + x(-e^{-x})$$

$$= e^{-x} - xe^{-x} = e^{-x}(1-x)$$

$\frac{dA}{dx} = 0$ when

$$e^{-x}(1-x) = 0$$

$$e^{-x} \neq 0$$

$$1-x = 0 \Rightarrow x = 1$$

	$x<1$	$x>1$
$\frac{dA}{dx}$:	+	−
	↗	↘

(at $x=1$)

$\Rightarrow x=1$ is a max

$$A(1) = 1(e^{-1}) = \boxed{\frac{1}{e}}$$

(Note: end pts: $x=0$: $A = 0e^0 = 0$

$x \to \infty$: $A = \lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{L'H}{=} \lim \ldots = 0$)

6. $m(t) = m_0 e^{kt}$

(a) $k \approx -1.2 \times 10^{-4} = -0.00012$

$\frac{1}{2}$-life: amount of time for half of initial mass to decay: $m(t_{1/2}) = m_0/2$

$$m(t_{1/2}) = \frac{m_0}{2} = m_0 e^{kt_{1/2}} \quad \left(= m_0 e^{-0.00012\, t_{1/2}}\right)$$

$$\Rightarrow \frac{1}{2} = e^{kt_{1/2}} \Rightarrow \ln\left(\frac{1}{2}\right) = k\, t_{1/2} \Rightarrow \frac{\ln(2)^{-1}}{k} = t_{1/2}$$

$$\Rightarrow \boxed{t_{1/2} = \frac{-\ln 2}{k} = \frac{\ln 2}{0.00012}}$$

(b) $m(t) = 15 = 100 e^{-0.00012t}$

$0.15 = e^{-0.00012t}$

$\ln 0.15 = -0.00012t$

$$\Rightarrow \boxed{t = \frac{\ln(0.15)}{-0.00012}}$$

$100g = m_0$

after 1 $\frac{1}{2}$-life: 50g left

after 2 $\frac{1}{2}$-lifes: 25g left

after 3 $\frac{1}{2}$-lifes: 12.5g left

So this is not quite 3 half-lives

7. (a) FALSE : $f(x) = 1$, $g(x) = \frac{x-1}{x-1}$ is a counter examp

(b) TRUE : Think of the integrals as area under f and g

(c) FALSE : $f(x) = x^2$ and $g(x) = 10$ on $[0,1]$ is an example

(d) TRUE : Continuity is a requirement for differentiability

(e) TRUE : $\log_{10}(x) = y \Leftrightarrow x = 10^y$ $\quad 1 = 10^y \Rightarrow y = 0$