Exam 3 Solutions:

F 8
T 16
T 24
F 32
16
8

1. a) $a_n > 0$ and $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

If $\sum_1^\infty a_n$ converges, the Limit Comparison Test is inconclusive.

so False

b) $\sum_2^\infty 5\left(\frac{1}{2}\right)^n = \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \ldots \quad a = \frac{5}{4} \quad r = \frac{1}{2}, \quad |r|<1$

so Geometric Series converges to $\frac{5/4}{1-1/2} = \frac{\frac{5}{4}}{\frac{1}{2}} = \boxed{\frac{5}{2}}$

True

c) $\sum_3^\infty ne^{-n^2}$ converges by the Integral Test:

$$\int_3^\infty ne^{-n^2}dn = \int_{\cancel{9}}^{\infty} e^{-u}\frac{du}{2} = \frac{1}{2}\int_9^\infty e^{-u}du$$

$u = +n^2$

$du = +2n\,dn$

$\frac{du}{+2n} = dn$

$$= \frac{1}{2}\lim_{b\to\infty}\left[-e^{-u}\right]_9^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[e^{-9} - e^{-b}\right]$$

$$= \frac{e^{-9}}{2} \text{ converges}$$

True

d) $\sum_5^\infty \ln\left[\frac{n}{n+1}\right] = \sum_5^\infty \ln[n] - \ln[n+1]$

$S_N = \sum_5^N \ln(n) - \ln(n+1) = \ln(5) - \cancel{\ln(6)} + \cancel{\ln(6)} - \cancel{\ln(7)} + \cdots + \cancel{\ln(N)} - \ln(N+1)$

$= \ln(5) - \ln(N+1)$

False

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln(5) - \ln(N+1) = -\infty$, DNE

2.a) $\sum_{7}^{\infty} \frac{1}{\sqrt{n^2-4}}$ Diverges DCT w/ $\frac{1}{n}$:

$0 \geq 4$

$n^2 \geq n^2-4$

$\frac{1}{n^2-4} > \frac{1}{n^2}$

$\frac{1}{\sqrt{n^2-4}} > \frac{1}{n} > 0$

$\sum^{\infty} \frac{1}{n}$ diverges

LCT w/ $\frac{1}{n}$: $\frac{\frac{1}{\sqrt{n^2-4}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2-4}} = \frac{1}{\sqrt{1-\frac{4}{n^2}}} \to 1$

Since $\sum_{7}^{\infty} \frac{1}{n}$ diverges, $\sum_{7}^{\infty} \frac{1}{\sqrt{n^2-4}}$ does too

b) $\sum_{11}^{\infty} \frac{4^{n+1}}{n^{10}3^n}$ RT: $\left(\frac{4^{n+2}}{(n+1)^{10}3^{n+1}}\right)\left(\frac{n^{10}3^n}{4^{n+1}}\right) = \frac{4}{3}\left(\frac{n}{n+1}\right)^{10} \to \frac{4}{3} > 1$

Diverges

NRT: $\left(\frac{4^{n+1}}{n^{10}3^n}\right)^{1/n} = \left(\frac{4^{1+\frac{1}{n}}}{n^{10/n}\,3}\right) = \frac{4}{3}\left(\frac{4^{1/n}}{(n^{1/n})^{10}}\right) \to \frac{4}{3} > 1$

c) $\sum_{13}^{\infty} \frac{\sin(n^2)}{n^{1.1}}$ $\underline{\text{ACT}}$ $\sum_{13}^{\infty} \frac{|\sin(n^2)|}{n^{1.1}}$ $\underline{\text{DCT}}$ to $\frac{1}{n^{1.1}}$

$0 < |\sin(n^2)| < 1 \Rightarrow 0 < \frac{|\sin(n^2)|}{n^{1.1}} < \frac{1}{n^{1.1}}$ and

$\sum_{13}^{\infty} \frac{1}{n^{1.1}}$ converges $p = 1.1 > 1$,

Converges (absolutely)

3\. $\sum_{17}^{\infty} \frac{(x-1)^{n+1}}{n2^n}$ is a Power Series. Start with ACT.

$\sum_{17}^{\infty} \left|\frac{(x-1)^{n+1}}{n2^n}\right|$ do Ratio Test:

$$\lim_{n\to\infty}\left(\frac{|x-1|^{n+2}}{(n+1)2^{n+1}}\right)\left(\frac{n2^n}{|x-1|^{n+1}}\right) = \lim_{n\to\infty} \frac{|x-1|}{2}\left(\frac{n}{n+1}\right) = \frac{|x-1|}{2} < 1$$

$$\frac{|x-1|}{2} < 1 \Rightarrow -1 < \frac{x-1}{2} < 1$$

$$\Rightarrow -2 < x-1 < 2$$

$$\Rightarrow -1 < x < 3$$ → series converges absolutely on this interval

a) Radius of Convergence is $\frac{3+1}{2} = \boxed{2}$

b) Check endpoints: @ $x=-1$, series is $\sum_{17}^{\infty} \frac{(-2)^{n+1}}{n2^n} = \sum_{17} \frac{(-1)^{n+1}\, 2^{n+1}}{n2^n}$

$= \sum_{17}^{\infty} \frac{2(-1)^{n+1}}{n}$ which is a conditionally convergent alternating series.

@ $x=3$, series is $\sum_{17}^{\infty} \frac{(2)^{n+1}}{n2^n} = \sum_{17}^{\infty} \frac{2}{n} = 2\sum_{17}^{\infty} \frac{1}{n}$

which is a divergent p-series with $p=1$.

IOC = $-1 \leq x < 3$

d) See work above. Converges conditionally at $x=-1$

c) See work above. Converges absolutely for $-1 < x < 3$

4.a) $\frac{1}{1+x^3} = \sum_0^{\infty} (-1)^n x^{3n}$ take derivative of both sides

$$\frac{-3x^2}{(1+x^3)^2} = \sum_1^{\infty} (-1)^n 3n x^{3n-1} = -3x^2 + 6x^5 - 9x^8 + \cdots$$

b) $\sum_{n=1}^{\infty} \frac{3n(-1)^n}{2^{3n-1}} = \sum_{n=1}^{\infty} (-1)^n 3n x^{3n-1} = \frac{-3x^2}{(1+x^3)^2}$ at $x = \frac{1}{2}$

$$= \frac{-3/4}{(1+1/8)^2} = -\frac{3}{4}\left(\frac{64}{81}\right) = \boxed{\frac{-16}{27}}$$

5. a) The Volume $= \frac{4}{3}\pi(1)^3 + \frac{4}{3}\pi\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{4}{3}\pi\left(\frac{1}{\sqrt{3}}\right)^3 + \cdots$

$= \frac{4\pi}{3}\left(1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots\right)$

yes, finite volume

$= \frac{4\pi}{3}\sum_{1}^{\infty}\frac{1}{n^{3/2}}$ converges: p-series $p = 3/2 > 1$

b) The Surface Area $= 4\pi(1)^2 + 4\pi\left(\frac{1}{\sqrt{2}}\right)^2 + 4\pi\left(\frac{1}{\sqrt{3}}\right)^2 + \cdots$

$= 4\pi\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$

no, infinite surface area

$= 4\pi\sum_{1}^{\infty}\frac{1}{n}$ diverges