Review#2(V.1) APPM1360 Name..

REVIEW #2 : CALCULUS II

Summary

7.1 Basic Integration Formulas

* *u-substitution* * *Completing the square*: $\sqrt{8x-x^2}=\sqrt{16-(x-4)^2}$

* *Trigonometric identities* :

$\sin^2 x+\cos^2 x=1,\ 1+\tan^2 x=\sec^2 x,\ 1+\cot^2 x=\csc^2 x,\ \sin 2x=2\sin x\cos x$

* *Eliminating a square root* : $\sqrt{1+\cos 4x}=\sqrt{2\cos^2 2x}=\sqrt{2}\,|\cos 2x|$

Need the formulas : $\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$

* *Reducing an improper fraction(long division)*

* *Separating a fraction* : $\dfrac{3x-2}{\sqrt{1-x^2}}=\dfrac{3x}{\sqrt{1-x^2}}-\dfrac{2}{\sqrt{1-x^2}}$

* *Multiply by a form of 1* : $\int \sec x\,dx=\int \sec x\cdot\dfrac{\sec x+\tan x}{\sec x+\tan x}dx$

7.2 Integration by Parts

* *Formula* : $\int u\,dv=uv-\int v\,du$ * *Tabular integration*

* *Solving for the unknown Integral* : $\int e^{2x}\sin x\,dx$

7.3 Partial Fractions

* *Write a rational function* $\dfrac{f(x)}{g(x)}$ *as a sum of partial fractions by regarding two things*

1. The degree of f(x) must be less than the degree of g(x). (If it isn't, do long division.)
2. Factor g(x) out as a product of real linear factors and real quadratic factors.

* *Example* : How to decompose a proper rational function $\dfrac{f(x)}{g(x)}$

1. $\dfrac{\text{numerator}}{(x+1)(x-3)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x-3)}$
2. $\dfrac{\text{numerator}}{(x+1)x^2(x-3)^3}=\dfrac{A}{(x+1)}+\dfrac{B}{x}+\dfrac{C}{x^2}+\dfrac{D}{(x-3)}+\dfrac{E}{(x-3)^2}+\dfrac{F}{(x-3)^3}$
3. $\dfrac{\text{numerator}}{(x+1)^2(2x^2+x+1)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+1)^2}+\dfrac{Cx+D}{(2x^2+x+1)}$
4. $\dfrac{\text{numerator}}{(x+1)^2(x^2+9)^3(2x^2+x+1)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+1)^2}+\dfrac{Cx+D}{(x^2+9)}+\dfrac{Ex+F}{(x^2+9)^2}+\dfrac{Gx+H}{(x^2+9)^3}+\dfrac{Ix+J}{(2x^2+x+1)}$

* *How to find the constants of partial fractions* :

1. Undetermined coefficients
2. Heaviside method for linear factors
3. Differentiation
4. Assigning numerical values to x

7.4 Trigonometric Substitutions

* *Three Basic Substitutions* :

1. To replace a^2+u^2, set $u=a\tan\theta$
2. To replace a^2-u^2, set $u=a\sin\theta$
3. To replace u^2-a^2, set $u=a\sec\theta$

7.6 Improper Integrals

* *Definition* : Integrals with infinite limits of integration and integrals of functions that become infinite at a point within the interval of integration are **improper integrals,** for example

$$\int_{-\infty}^{1} x^2dx,\ \int_{-\infty}^{\infty}\frac{x}{x^2+1}dx,\ \int_{1}^{5}\frac{1}{x-1}dx,\ \int_{0}^{3}\ln x\,dx,\ \int_{0}^{10}\frac{1}{x^2+x-2}dx,\ \int_{0}^{\infty}\frac{1}{x(x-1)}dx.$$

* *Definition of Types of Improper integrals* :

1. If f is continuous on $[a,\infty)$, then $\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$.

2. If f is continuous on $(-\infty,b]$, then $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$.

3. If f is continuous on $(a,b]$, then $\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$.

4. If f is continuous on $[a,b)$, then $\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$.

In each case, if the limit is finite then the improper integral **converges** and the limit is the value of the improper integral. If the limit fails to exist then the improper integral **diverges.**

* *Definition* : If f becomes infinite at an interior point $d\in[a,b]$, then $\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$. The integral from a to b **converges** if the integrals from a to d and d to b both converge. Otherwise, the integral from a to b **diverges.**

* *Definition* : If f is continuous on $(-\infty,\infty)$ and if $\int_{-\infty}^{a} f(x)dx$ and $\int_a^{\infty} f(x)dx$ both converge, then $\int_{-\infty}^{\infty} f(x)dx$ **converges** and define its value to be $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$.

If either or both of the integrals on the right-hand side of above equation diverge, then $\int_{-\infty}^{\infty} f(x)dx$ **diverges.**

* *Remark* : 1. $\int_a^{\infty}\frac{1}{x^p}dx, a\ge 1$ converges if $p>1$ and diverges if $p\le 1$.

2. $\int_0^{b}\frac{1}{x^p}dx$ (b is a finite number.) converges if $p<1$ and diverges if $p\ge 1$.

3. For any number p, $\int_0^{\infty}\frac{1}{x^p}dx$ is always divergent.

* *Tests for Convergence and Divergence of improper integrals* :

Direct Comparison Test (DCT) : Let f and g be continuous on $[a,\infty)$ and suppose that $0\le f(x)\le g(x)$ for all $x\ge a$. Then

13. $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx$ 14. $\int_0^\infty \frac{1}{x^p}dx$, for any real number p 15. $\int_{-\infty}^\infty \frac{1+x}{1+x^2}dx$

16. Evaluate the limit: $\lim_{t\to\infty}\int_{-t}^{t}\frac{1+x}{1+x^2}dx$

b) ***Without evaluation****, determine whether the following integrals converge or diverge. Besure to justify your answer.*

1. $\int_1^\infty \frac{1-\cos x}{x^3}dx$ 2. $\int_{-\pi/2}^{\pi/2}\frac{\cos\theta}{(\pi-2\theta)^{1/3}}d\theta$ 3. $\int_\pi^\infty \frac{1+\sin x}{x^2}dx$ 4. $\int_1^\infty \frac{\sin^2 t}{t^2}dt$

5. $\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}dt$ 6. $\int_2^\infty \frac{dx}{\ln x}$ 7. $\int_{-1}^{1}\ln|x|dx$ 8. $\int_0^\infty (x^2+x^4)^{-\frac{1}{3}}dx$

9. $\int_0^1 \frac{1}{\sqrt{x}}\sin\left(\frac{1}{x}\right)dx$ 10. $\int_5^\infty \frac{s+1}{\sqrt{s^4+3s}}ds$ 11. $\int_1^\infty \frac{1}{x^3}\sqrt{1-\frac{1}{x^6}}dx$ 12. $\int_0^1 \frac{dt}{t-\sin t}$

13. $\int_0^\infty \frac{1}{\sqrt{x+x^3}}dx$ 14. $\int_0^{\pi/2}\sqrt{\theta}\cot\theta\, d\theta$ 15. $\int_{-\infty}^\infty \frac{x}{1+x^2}dx$ 16. $\int_1^\infty \frac{dx}{e^x-e^{-x}}$

17. $\int_1^\infty \frac{dx}{e^x-2^x}$ 18. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$ 19. $\int_0^\infty \frac{\sin^2 x}{x^{2.5}}dx$ 20. $\int_0^\infty \frac{\sin(x^2)}{x^{2.5}}dx$

III. Sequences and their convergence and divergence

a) Determine whether the following sequences converge or diverge. Give the limits if they exist.

1. $a_n=\sqrt{\frac{2n}{n+1}}$ 2. $a_n=\frac{n}{\sqrt{n^2+3n+4}}$ 3. $a_n=\frac{3^n}{n^3}$ 4. $a_n=\frac{(\ln n)^2}{n}$

5. $\{\ln n-\ln(n+1)\}_{n=1}^\infty$ 6. $a_n=\ln(n^2+1)-\ln(n^2)$ 7. $\left\{\cos\left(\frac{3\pi n^2-n}{9n^2+1}\right)\right\}_{n=1}^\infty$ 8. $a_n=\sin\left(n\pi-\frac{1}{n}\right)$

9. $a_n=\frac{1}{n}\int_1^n \frac{1}{x}dx$ 10. $a_n=n\left(1-\cos\frac{1}{n}\right)$ 11. $a_n=\left(1+\frac{3}{n^2}\right)^{\frac{1}{n}}$ 12. $a_n=\sinh(\ln n)$

13. $\{n^n e^{-2n}\}_{n=1}^\infty$ 14. $a_n=(-1)^n$ 15. $a_n=(-1)^n\left(1-\frac{1}{n}\right)$ 16. $a_n=\cos\left(\frac{n\pi}{4}\right)$

17. $a_n=e^{-n}\cos(n\pi)$ 18. $\left\{\cos(n\pi)\cdot\left(\frac{1}{n}\right)^n\right\}_{n=1}^\infty$ 19. $a_n=\frac{n+n(-1)^n}{n!}$ 20. $\left\{\frac{\sin n}{\sqrt{n}}\right\}_{n=1}^\infty$

21. $a_n=\frac{\sin n}{n^2}$ 22. $a_n=\frac{\sin n!}{\sqrt{n+1}}$ 23. $a_n=n\left(1-\sqrt{1-\frac{1}{n}}\right)$ 24. $a_n=\frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$

25. $a_n=\sqrt{n+\sqrt{n}}-\sqrt{n}$ 26. $a_n=\frac{n!}{2^n}$ 27. $a_n=\left(1-\frac{1}{n^2}\right)^n$ 28. $a_n=\left(\frac{n-2}{n}\right)^n$

29. $a_n = \left(\frac{n}{n-1}\right)^n$ 30. $a_n = \left(n^2+n\right)^{1/n}$ 31. $a_n = n^{(2/\ln n)}$ 32. $0 < a_n < 2, a_n < a_{n+1}$

33. $\{a_n\}_{n=1}^{\infty}$ where for all $n \geq 1, a_{2n} = 0, a_{2n+1} > a_{2n-1}, a_1 = 1$ and $a_n < 2$.

b) Let $I_n = \int_0^1 (\ln x)^n dx, n \geq 0$.

1). Find an equation relating I_n and I_{n-1}

2). It can be shown that $I_n = (-1)^n n!$. Determine whether the sequence $\{I_n\}_{n=1}^{\infty}$ converges or diverges.

c) Let $a_n = \left(1-\frac{2}{n}\right)^n$. Does the sequence $\{a_n\}$ converge? Does the series $\sum_{n=1}^{\infty} a_n$ converge?

d) Let $a_n = \left(\frac{3^n}{2n+1}\right)^{\frac{1}{n}}$. Does the sequence $\{a_n\}$ converge? Does the series $\sum_{n=1}^{\infty} a_n$ converge?

e) Let $\{F_n\}$ be the Fibonacci sequence defined as $F_1 = 1, F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Given the limit $\tau = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ exists show that $\tau = (1+\sqrt{5})/2$.

IV. Series and their convergence and divergence

a) Determine if each series below converges or diverges. If possible, for each convergent series determine the infinite sum of the series.

1. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$ 2. $\sum_{n=1}^{\infty} \cos(\pi n)$ 3. $\sum_{n=0}^{\infty} \left(1-\frac{2}{n}\right)^n$ 4. $\sum_{n=3}^{\infty} \left(1-\frac{1}{2n}\right)^n$

5. $\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}}$ 6. $\sum_{n=0}^{\infty} \frac{2^n+1}{3^{2n}}$ 7. $\sum_{n=0}^{\infty} \frac{2^n-1}{3^n}$ 8. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n}$

9. $\sum_{n=3}^{\infty} (-1)^{n+1}\frac{3}{2^n}$ 10. $\sum_{n=1}^{\infty} e^{-n}$ 11. $3+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ 12. $10+\frac{10}{3}+\frac{10}{9}+\ldots$

13. $0.4+0.16+0.064+0.0256+\ldots$ 14. $(1+\ln 2)-(1+\ln 2)^2+(1+\ln 2)^3-(1+\ln 2)^4+\ldots$

15. $\sum_{n=0}^{\infty} (-1)^n x^{2n}, x = \text{constant}$ 16. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ 17. $\sum_{n=1}^{\infty} \left(\frac{2}{2n+1}-\frac{2}{2n+3}\right)$ 18. $\sum_{n=0}^{\infty} \frac{3}{(n+1)(n+3)}$

19. $\frac{2}{3\cdot4\cdot5}+\frac{2}{4\cdot5\cdot6}+\frac{2}{5\cdot6\cdot7}+\frac{2}{6\cdot7\cdot8}+\ldots$ 20. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ 21. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

22. $\sum_{n=1}^{\infty} \left\{\frac{\cos n}{n^2}-\frac{\cos(n+1)}{(n+1)^2}\right\}$ 23. $\sum_{n=1}^{\infty} \tan^{-1}(n-1)-\tan^{-1}(n)$ 24. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ 25. $\sum_{n=3}^{\infty} \frac{1}{n\ln n}$

26. $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$ 27. $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ 28. $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\ldots+n}$

GEEN 1360 (Calculus II)
Methods for Determining Convergence of Series

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1. $\mathbf{n^{th}}$ - term test for **divergence**

   - $\sum_{n=0}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$

   - $\lim_{n\to\infty} a_n = 0$ is inconclusive

2. Geometric Series : $\sum_{n=0}^{\infty} a \cdot r^n$

   - converges to $\frac{a}{1-r}$ if $|r| < 1$

   - diverges if $|r| \geq 1$

3. **p** - series test : $\sum_{n=1}^{\infty} \frac{1}{n^p}$

   - converges if $p > 1$

   - diverges if $p \leq 1$

4. Telescoping Series : $\sum_{n=1}^{\infty} f(n) - f(n+1) = f(1) - \lim_{n\to\infty} f(n+1) = L$

   - converges if L is finite

   - diverges if L is infinite

5. Integral Test : $\sum_{n=N}^{\infty} a_n = f(n)$ where f is continuous, positive and decreasing for all $x \geq N$

   - converges if $\int_N^{\infty} f(x)\,dx$ converges

   - diverges if $\int_N^{\infty} f(x)\,dx$ diverges

6. Direct Comparison Test : $\sum_{n}^{\infty} a_n$ has no negative terms

   - if $a_n \leq c_n$ and $\sum c_n$ converges then $\sum a_n$ also converges

   - if $d_n \leq a_n$ and $\sum d_n$ diverges then $\sum a_n$ also diverges

7. Limit Comparison Test : $a_n > 0$ and $b_n > 0$ for all $n \geq N$

- if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge
- if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ also converges
- if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ also diverges

8. Ratio Test : $\sum a_n$ has positive terms and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = p$

- converges if $p < 1$
- diverges if $p > 1$
- inconclusive if $p = 1$

9. $\mathbf{n^{th}}$ - root test: $\sum a_n$ has no negative terms for some $n \geq N$ and $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = p$

- converges if $p < 1$
- diverges if $p > 1$
- inconclusive if $p = 1$

10. Alternating Series Test (Leibniz's Theorem): $\sum\limits_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ converges if and only if

- u_n's are all positive
- $u_n \geq u_{n+1}$ for all $n \geq N$ where N is an integer
- $u_n \to 0$

11. Absolute Convergence Test

- if $\sum\limits_{n=1}^{\infty} |a_n|$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely
- a series that converges, but does not converge absolutely, converges conditionally

12. Method Flowchart on p 660