APPM 1360 | EXAM II | SPRING 2009

① (a) $$\int \sqrt{4-9t^2}\, dt = \int 2\sin\theta \cdot \tfrac{2}{3}(-\sin\theta)\, d\theta = -\tfrac{4}{3}\int \sin^2\theta\, d\theta$$

$$\left(\begin{array}{l} t = \tfrac{2}{3}\cos\theta \\ 4-9t^2 = 4-4\cos^2\theta = 4\sin^2\theta \\ \sqrt{4-9t^2} = 2\sin\theta \\ dt = \tfrac{2}{3}(-\sin\theta)\, d\theta \end{array}\right) \qquad \left(\begin{array}{l} \cos(2\theta) = 1-2\sin^2\theta \\ \sin^2\theta = \dfrac{1-\cos(2\theta)}{2} \end{array}\right)$$

$$= -\tfrac{4}{3}\int \frac{1-\cos 2\theta}{2}\, d\theta = -\tfrac{4}{3}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C =$$

$$= \tfrac{1}{3}\sin(2\theta) - \tfrac{2}{3}\theta + C \qquad \left(\begin{array}{l} \cos\theta = \dfrac{3t}{2} \Rightarrow \sin\theta = \sqrt{1-\dfrac{9t^2}{4}} = \dfrac{\sqrt{4-9t^2}}{2} \\ \theta = \cos^{-1}\left(\dfrac{3t}{2}\right) \end{array}\right)$$

$$= \tfrac{2}{3}\sin\theta\cos\theta - \tfrac{2}{3}\theta + C$$

$$= \cancel{\tfrac{2}{3}} \cdot \frac{\cancel{3}t}{\cancel{2}} \cdot \frac{\sqrt{4-9t^2}}{2} - \tfrac{2}{3}\cos^{-1}\left(\frac{3t}{2}\right) + C = \boxed{\tfrac{1}{2}t\sqrt{4-9t^2} - \tfrac{2}{3}\cos^{-1}\left(\tfrac{3t}{2}\right) + C}$$

(b) $$\int \frac{1}{(2x-3)(2x-1)}\, dx =$$

$$\frac{1}{(2x-3)(2x-1)} = \frac{A}{2x-3} + \frac{B}{2x-1} = \frac{A(2x-1)+B(2x-3)}{(2x-3)(2x-1)} \Rightarrow$$

$$\cancel{(2A+2B)x - A - 3B = 1} \Rightarrow A(2x-1) + B(2x-3) = 1$$

$$x = \tfrac{1}{2} \Rightarrow 0A - 2B = 1 \Rightarrow B = -\tfrac{1}{2}$$

$$x = \tfrac{3}{2} \Rightarrow 2A + 0B = 1 \Rightarrow A = \tfrac{1}{2}$$

$$= \int\left(\frac{1/2}{2x-3} + \frac{-1/2}{2x-1}\right) dx = \tfrac{1}{2}\cdot\int \frac{1}{2x-3}\, dx - \tfrac{1}{2}\int \frac{1}{2x-1}\, dx$$

$$= \tfrac{1}{2}\,\frac{\ln|2x-3|}{2} - \tfrac{1}{2}\,\frac{\ln|2x-1|}{2} + C = \tfrac{1}{4}\left(\ln|2x-3| - \ln|2x-1|\right) + C =$$

$$= \boxed{\frac{1}{4} \ln \left| \frac{2x-3}{2x-1} \right| + C}$$

(c) $\int_0^\infty x e^{-x}\, dx = \lim_{b\to\infty} \int_0^b x e^{-x}\, dx = \lim_{b\to\infty} \left(-x e^{-x} \Big|_0^b - \int_0^b -e^{-x}\, dx \right) =$

$$\left(\begin{cases} u = x \\ dv = e^{-x} dx \end{cases} \Rightarrow \begin{matrix} du = dx \\ v = -e^{-x} \end{matrix} \right)$$

$$= \lim_{b\to\infty} \left(-be^{-b} + \int_0^b e^{-x}\, dx \right) = \lim_{b\to\infty} \left(-be^{-b} - e^{-x}\Big|_0^b \right) =$$

$$= \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1 \right) = -0 - 0 + 1 = \boxed{1}$$

The integral is convergent.

$$\left(\begin{matrix} \lim_{b\to\infty} e^{-b} = 0 \\ \lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \frac{b}{e^b} \overset{\text{L'Hôpital}}{=\!=} \lim_{b\to\infty} \frac{1}{e^b} = 0 \end{matrix} \right)$$

(2) (a) $\frac{\left(\frac{4}{5}\right)^n}{\left(\frac{3}{4}\right)^n} = \left(\frac{4}{5}\right)^n \cdot \left(\frac{4}{3}\right)^n = \left(\frac{16}{15}\right)^n \longrightarrow \infty$ because $\frac{16}{15} > 1$

(the sequence diverges)

(b) $2 \le x < \infty \Rightarrow x^2 > x \Rightarrow -x^2 < -x \Rightarrow 0 < e^{-x^2} < e^{-x}$

Solution I

$\int_2^\infty e^{-x}\, dx = \lim_{b\to\infty} \left(-e^{-x} \Big|_2^b \right) = \lim_{b\to\infty} \left(-e^{-b} + e^{-2} \right) = e^{-2} \Rightarrow$

$\Rightarrow \int_2^\infty e^{-x}\, dx$ converges

By the direct comparison criterion : $\int_2^\infty e^{-x^2}\, dx$ converges.

(b) Consider the series $\sum_{n=2}^{\infty} e^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{e}\right)^n$ convergent as a geometric series with $r = \frac{1}{e} < 1$

Solution I

For $n \geq 2$: $0 < e^{-n^2} < e^{-n} \Rightarrow \sum_{n=2}^{\infty} e^{-n^2}$ convergent, by the direct comparison criterion for series.

The function $f(x) = e^{-x^2}$ decreases to 0 as $x \to \infty$, and $f(n) = e^{-n^2}$

By Integral Comparison $\Rightarrow \int_2^{\infty} e^{-x^2} dx$ convergent.

(c) $a_n = \left(\frac{n}{3n+4}\right)^n$

$\ln(a_n) = n \ln\left(\frac{n}{3n+4}\right)$

$\lim_{n\to\infty} \frac{n}{3n+4} = \frac{1}{3} \Rightarrow \lim_{n\to\infty} \ln\left(\frac{n}{3n+4}\right) = \ln\frac{1}{3} = -\ln 3 \Rightarrow$

$\lim_{n\to\infty} n \cdot \ln\left(\frac{n}{3n+4}\right) = \infty \cdot (-3) = -\infty \Rightarrow a_n \to e^{-\infty} = 0$

Divergence criterion does not apply

$a_n > 0$ for all n. We apply the n-th Root Test:

$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n}{3n+4}\right)^n} = \frac{n}{3n+4} = \frac{1}{3} < 1 \Rightarrow$ series converges.

(3) (a) $\sum_1^{\infty} \frac{1+4^{n-2}}{4^n} = \sum_1^{\infty} \left(\frac{1}{4}\right)^n + \frac{1}{4^2}$

$a_n = \left(\frac{1}{4}\right)^n + \frac{1}{16} \longrightarrow \frac{1}{16}$ as $n \to \infty$

Divergence criterion $\Rightarrow$ series is divergent, since $a_n \not\to 0$.

(b) Using the similar result in problem 1(b), we get:

$$\frac{1}{(2n-3)(2n-1)} = \frac{1}{2}\cdot\frac{1}{2n-3} - \frac{1}{2}\cdot\frac{1}{2n-1}$$

$$\sum_{2}^{\infty} \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-1} + \cdots\right)$$

$$s_n = \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{2n-3}} - \frac{1}{2n-1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n-1}\right)$$

$$\lim_{n\to\infty} s_n = \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}$$

The series is a convergent telescoping series, ~~which~~ whose sum is $\boxed{\frac{1}{2}}$

(c) $f(x) = \dfrac{1}{x(\ln x)^3}$ positive, decreasing to zero for $x \to \infty$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{x}\cdot\frac{1}{u^3}\, x\,du = \lim_{b\to\infty} \left.\frac{u^{-2}}{-2}\right|_{\ln 2}^{\ln b}$$

$$\left(\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \implies dx = x\,du \\ u(2) = \ln 2 \\ u(b) \end{array}\right)$$

$$= \lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}.$$

Integral is convergent. By the Integral Test $\Rightarrow \sum_{2}^{\infty} f(n) = \sum_{2}^{\infty} \frac{1}{n(\ln n)^3}$ converges.

(d) $a_n = \dfrac{n^4 \cdot n!}{(2n)! \cdot 3^n} > 0$ for all $n \geq 1$

Ratio Test: $$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4 \cdot (n+1)!}{\underbrace{(2n+2)!}_{(2n+1)(2n+2)} \cdot 3^{n+1}} \cdot \frac{(2n)! \cdot 3^n}{n^4 \cdot n!} =$$

$$= \frac{(n+1)^4 \cdot (n+1)}{n^4 \cdot (2n+1)(2n+2)} = \left(\frac{n+1}{n}\right)^4 \cdot \frac{n+1}{(2n+1)(2n+2)} \longrightarrow 1 \cdot 0 = 0 < 1$$

By the Ratio Test for positive series $\Longrightarrow$ series converges.

(4) (a) FALSE. Counterexample: $\frac{1}{n} \longrightarrow 0$, but $\sum_1^\infty \frac{1}{n}$ diverges.

(b) FALSE. $\sum_1^\infty \frac{3}{2^n} = 3 \cdot \left(\sum_0^\infty \left(\frac{1}{2}\right)^n - 1 \right) = 3 \cdot \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = \boxed{3} \neq 6$

(c) FALSE.

$$\frac{a_{n+1}}{a_n} = \frac{\ln^4(n+1)}{n+1} \cdot \frac{n}{\ln^4 n} = \frac{n}{n+1} \cdot \left(\frac{\ln(n+1)}{\ln n} \right)^4$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\text{L'Hôpital}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$$

Hence: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow$ Ratio Test is inconclusive.

(d) ~~FALSE~~. For $n \geq 1 \Rightarrow e^{-n} < e^{-1} < 1 \Rightarrow \frac{e^{-n}}{n} < \frac{1}{n}$

However: $\sum \frac{1}{n}$ diverges, so the test is inconclusive.

(e) TRUE.