

1. (16 pts) Do the following sequences converge or diverge? If convergent, give the limit. Explicitly show your reasoning.

(a)  $a_n = (-1)^n \frac{n}{n+1}$       (b)  $\left\{ (-1)^n \frac{n}{n^2+1} \right\}_{n \geq 22}$

**Solution:**

(a) Sequence diverges since  $\frac{n}{n+1} \rightarrow 1$  as  $n \rightarrow \infty$  so  $a_n = (-1)^n \frac{n}{n+1}$  oscillates between -1 and 1.

(b) Sequence converges to 0 by Squeeze Theorem since,

$$\frac{-n}{n^2+1} \leq \frac{(-1)^n n}{n^2+1} \leq \frac{n}{n^2+1} \text{ for } n \geq 1$$

and  $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = \lim_{n \rightarrow \infty} \frac{-n}{n^2+1} \underbrace{=}_{L'H} 0.$

2. (20 pts) For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answers.

(a)  $\sum_{n=10}^{\infty} \frac{\cos(n\pi)}{n^{3/2}-1}$       (b)  $\sum_{n=21}^{\infty} \frac{(-1)^n}{\ln(n)+n}$       (c)  $\sum_{n=88}^{\infty} ne^{-n}$

**Solution:**

(a) Series is absolutely convergent since

$$\sum_{n=10}^{\infty} \left| \frac{\cos(n\pi)}{n^{3/2}-1} \right| = \sum_{n=10}^{\infty} \left| \frac{(-1)^n}{n^{3/2}-1} \right| = \sum_{n=10}^{\infty} \frac{1}{n^{3/2}-1}$$

and  $\sum_{n=10}^{\infty} \frac{1}{n^{3/2}-1}$  converges by Limit Comparison Test with  $\sum_{n=10}^{\infty} \frac{1}{n^{3/2}}$ , a convergent p-series.

(b) Series is conditionally convergent since for  $n \geq 1$  we have  $\ln(n) < n$  implies  $\ln(n) + n < 2n$ , and

$$\sum_{n=21}^{\infty} \left| \frac{(-1)^n}{\ln(n)+n} \right| = \sum_{n=21}^{\infty} \frac{1}{\ln(n)+n} > \sum_{n=21}^{\infty} \frac{1}{2n}$$

so  $\sum_{n=21}^{\infty} \frac{1}{\ln(n)+n}$  diverges by Direct Comparison with  $\sum_{n=21}^{\infty} \frac{1}{2n}$  which is divergent (since it is a nonzero

multiple of a divergent harmonic series.) The alternating series  $\sum_{n=21}^{\infty} \frac{(-1)^n}{\ln(n)+n}$  converges by the Alter-

nating Series Test since  $u_n = \frac{1}{\ln(n)+n}$  and note that  $u_{n+1} = \frac{1}{\ln(n+1)+(n+1)} < \frac{1}{\ln(n)+n} = u_n$  and  $\lim_{n \rightarrow \infty} u_n = 0.$

(c) Absolutely convergent by Ratio (or Root) test since,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{1}{e} = \frac{1}{e} < 1$$

3. (20 pts) Show all work.

(a) Use series to evaluate:  $\lim_{x \rightarrow 0} \frac{\sin(x) - \tan^{-1}(x)}{x^3}$ .

(b) Find the **Taylor Polynomial of order 2** generated by  $f(x) = (1-x)^{1/2}$  at  $x = 0$ .

**Solution:**

(a) Converges to  $1/6$  since,

$$\lim_{x \rightarrow 0} \frac{\sin(x) - \tan^{-1}(x)}{x^3} = \lim_{x \rightarrow 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots) - (x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots)}{x^3}, \text{ and moreover}$$

$$\lim_{x \rightarrow 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots) - (x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots)}{x^3} = \lim_{x \rightarrow 0} \frac{(\frac{x^3}{3} - \frac{x^3}{3!}) + (\frac{x^5}{5!} - \frac{x^5}{5}) + (\frac{x^7}{7} - \frac{x^7}{7!}) + \dots}{x^3}, \text{ and finally}$$

$$\lim_{x \rightarrow 0} \frac{(\frac{x^3}{3} - \frac{x^3}{3!}) + (\frac{x^5}{5!} - \frac{x^5}{5}) + (\frac{x^7}{7} - \frac{x^7}{7!}) + \dots}{x^3} = \lim_{x \rightarrow 0} \frac{x^3 \left( (\frac{1}{3} - \frac{1}{3!}) + (\frac{x^2}{5!} - \frac{x^2}{5}) + (\frac{x^4}{7} - \frac{x^4}{7!}) + \dots \right)}{x^3} = \frac{1}{3} - \frac{1}{3!} = \frac{1}{6}$$

(b) Note,

$$(1-x)^{1/2} = 1 + \sum_{k=1}^{\infty} \binom{1/2}{k} (-x)^k = 1 - \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \dots = 1 - \frac{x}{2} + \frac{(1/2)(-1/2)}{2!}x^2 + \dots$$

$$\text{so } P_2(x) = 1 - \frac{x}{2} - \frac{x^2}{8}.$$


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4. (24 pts) Justify your answers.

(a) Find a Maclaurin Series for  $f(x) = xe^{-x/2}$ . (You may use your knowledge of the Maclaurin Series of  $e^x$  to answer this question.)

(b) Use the first 3 non-zero terms of the series found in part (a) to approximate  $\int_0^1 xe^{-x/2} dx$ .

(c) Estimate the error of the approximation found in part (b).

(d) Is the approximation found in part (b) an *underestimate* or an *overestimate*?

**Solution:**

$$(a) xe^{-x/2} = x \cdot \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^n n!} = x - \frac{x^2}{2} + \frac{x^3}{2^2 2!} - \frac{x^4}{2^3 3!} + \dots$$

$$(b) \int_0^1 xe^{-x/2} dx \approx \int_0^1 \left( x - \frac{x^2}{2} + \frac{x^3}{8} \right) dx = \left( \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{32} \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{6} + \frac{1}{32}$$

$$(c) \int_0^1 xe^{-x/2} dx = \frac{1}{2} - \frac{1}{6} + \frac{1}{32} - \frac{1}{5 \cdot 2^3 \cdot 3!} + \dots, \text{ so } |\text{Error}| \leq \left| -\frac{1}{5 \cdot 2^3 \cdot 3!} \right| = \frac{1}{240}$$

(One could also use the Taylor Remainder Term  $R_3(x)$  to estimate the error.)

(d) Note,  $-\frac{1}{5 \cdot 2^3 \cdot 3!} < 0$  implies the approximation used in part (b) is an overestimate.

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5. (20 pts) Show all work.

(a) Find the interval of convergence of  $\sum_{n=0}^{\infty} (n+1)x^n$ .

(b) For what values of  $x$  is the series conditionally convergent? absolutely convergent? divergent?

(c) Show that the series in part (a) converges to  $\frac{1}{(1-x)^2}$ .

(d) Find the sum of the series  $\sum_{n=0}^{\infty} \frac{n+1}{3^n}$ .

**Solution:**

(a) Using Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = |x| \cdot \lim_{n \rightarrow \infty} \frac{n+2}{n+1} = |x|$$

so letting  $|x| < 1$  implies  $-1 < x < 1$  and when  $x = 1$  we have  $\sum_{n=0}^{\infty} (n+1)$  which diverges by

the  $n^{\text{th}}$  Term Divergence Test, and if  $x = -1$  we have  $\sum_{n=0}^{\infty} (-1)^n (n+1)$  which also diverges by the  $n^{\text{th}}$  Term Divergence Test, so the Interval of Convergence is  $(-1, 1)$ .

(b) (i) Not conditionally convergent for any values of  $x$ .

(ii) Absolutely convergent for  $-1 < x < 1$ .

(iii) Divergent on  $(-\infty, -1] \cup [1, \infty)$

$$(c) \frac{1}{(1-x)^2} = \frac{d}{dx} \left[ \frac{1}{1-x} \right] = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} x^n \right] = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} (n+1)x^n$$

(One could also use a Binomial Series expansion here.)

(d) Series converges to  $9/4$  since

$$\sum_{n=0}^{\infty} \frac{n+1}{3^n} = \sum_{n=0}^{\infty} (n+1)x^n \Big|_{x=1/3} = \frac{1}{(1-x)^2} \Big|_{x=1/3} = \frac{1}{(1-1/3)^2} = 9/4.$$

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