7.4 Trigonometric Substitutions

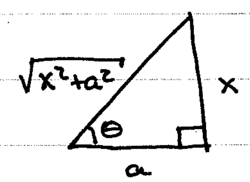
3 main types

	type	sub
1.	$a^2 + x^2$	$x = a\tan\theta$
2.	$a^2 - x^2$	$x = a\sin\theta$
3.	$x^2 - a^2$	$x = a\sec\theta$

3 triangles used after Integration

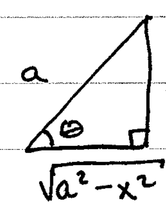
1. $x = a\tan\theta$

$\Rightarrow \tan\theta = \frac{x}{a}$



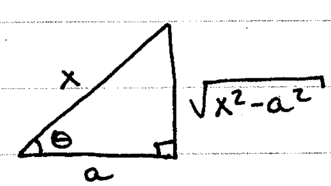
2. $x = a\sin\theta$

$\Rightarrow \sin\theta = \frac{x}{a}$



3. $x = a\sec\theta$

$\Rightarrow \sec\theta = \frac{x}{a}$



Pythag. Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

#4

Ex: $\displaystyle\int_0^2 \frac{dx}{8+2x^2}$

$= \displaystyle\frac{1}{2}\int_0^2 \frac{dx}{4+x^2}$

$a = 2$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

Endpoints

$x = 2 \Rightarrow \theta = \frac{\pi}{4}$

$x = 0 \Rightarrow \theta = 0$

$= \displaystyle\frac{1}{2}\int_0^{\pi/4} \frac{2\sec^2\theta \, d\theta}{2^2 + 2^2\tan^2\theta}$

$= \displaystyle\frac{1}{8}\int_0^{\pi/4} \frac{2\sec^2\theta \, d\theta}{1+\tan^2\theta}$

$= \displaystyle\frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{\cancel{\sec^2\theta} \, d\theta}{\cancel{\sec^2\theta}}$

$= \displaystyle\frac{1}{4}\int_0^{\pi/4} d\theta$

$= \displaystyle\frac{\theta}{4}\Big|_0^{\pi/4}$

$= \boxed{\frac{\pi}{16}}$

#18

Ex: $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\, dw$

No Endpts ⇒ use △ at end to go back to w.

$a = 3$

$w = 3\sin\theta$

$dw = 3\cos\theta$

$$\int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta \, d\theta$$

all the constants cancel

$$\int \frac{\sqrt{1-\sin^2\theta} \cdot \cos\theta}{\sin^2\theta}\, d\theta$$

$$= \int \frac{\sqrt{\cos^2\theta}\;\cos\theta}{\sin^2\theta}\, d\theta$$

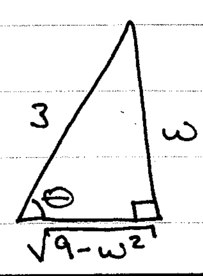
$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta \, d\theta = 9\int (\csc^2\theta - 1)\, d\theta$$

$$= \left[-\cot\theta - \theta\right]$$

go back to w

$$= \boxed{\sqrt{9-w^2} - \sin^{-1}\left(\frac{w}{3}\right) + C}$$



#31

Ex: $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

$= \displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}(1+4t)}$

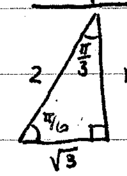
$u = \sqrt{t}$

$du = \dfrac{1}{2\sqrt{t}}\,dt$

$2du = \dfrac{dt}{\sqrt{t}}$

$\displaystyle\int_{1/\sqrt{12}}^{1/2} \frac{4\,du}{1+4u^2}$

Endpoints



$u = \frac{1}{2}\tan\theta$

$2u = \tan\theta$

$= \displaystyle\int_{1/2\sqrt{3}}^{1/2} \frac{du}{\frac{1}{4}+u^2}$

$a = \frac{1}{2}$

$u = \frac{1}{2}\tan\theta$

$du = \frac{1}{2}\sec^2\theta\,d\theta$

$= \displaystyle\frac{1}{2}\int_{\pi/6}^{\pi/4} \frac{\sec^2\theta\,d\theta}{\frac{1}{4}+\frac{1}{4}\tan^2\theta}$

$= 2\displaystyle\int_{\pi/6}^{\pi/4} \frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} = 2\int_{\pi/6}^{\pi/4} d\theta$

$= 2\theta\Big|_{\pi/6}^{\pi/4} = 2\left[\frac{\pi}{4}-\frac{\pi}{6}\right] = \frac{\pi}{2}-\frac{\pi}{3} = \boxed{\frac{\pi}{6}}$

Ex: #27

$$\int \frac{v^2\,dv}{(1-v^2)^{5/2}} \qquad\qquad v = \sin\theta$$

$$dv = \cos\theta\,d\theta$$

$$= \int \frac{\sin^2\theta\,d\theta}{(1-\sin^2\theta)^{5/2}} \cdot \cos\theta\,d\theta$$

$$= \int \frac{\sin^2\theta \cdot \cos\theta\,d\theta}{(\cos^2\theta)^{5/2}}$$

$$= \int \frac{\sin^2\theta \cdot \cos\theta\,d\theta}{\cos^5\theta}$$

$$= \int \frac{\sin^2\theta\,d\theta}{\cos^4\theta}$$

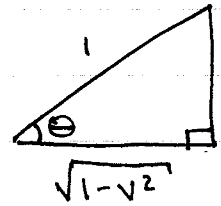
$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$= \int \tan\theta^2 \sec\theta^2 d\theta = \int u^2 du = \frac{u^3}{3} + c$$

$$= \frac{\tan^3\theta}{3} + c$$



$$= \boxed{\frac{v}{\sqrt{1-v^2}} + C}$$