1360 Lecture Friday 10-2-09

* Very Exciting!! Next Time Testing for Convergence!!

* Next wed. begin ch. 8 - Seq. & Series... Read Ahead

7.6 Improper Integrals (Part 1 of 2)

So far... only able to integrate functions with

1. Finite Domain AND that are
2. Finite on their domain

Now, will learn how to deal with Infinite domains & ~~infinite~~ fns that $\to \pm\infty$ on the domain.

Review of Limits

Ex: $\lim_{x\to\infty} \frac{2x^2+x}{5x^2-1} = \frac{2}{5}$ dominant powers =

$\lim_{x\to\infty} \frac{2x^2+x}{3x^4-7x^2+1} = 0$ deg bottom > deg top

$\lim_{x\to\infty} \frac{4x^4+7x-2}{3x^4-9} = \infty$ deg top > deg bottom

* Review L & R hand limits

Ex: $\lim_{x\to 1} \frac{1}{x-1} = \text{DNE}$

b/c

$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

$\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$

$L \neq R \Rightarrow \lim_{x\to 1}$ DNE

* In Integration L & R limits ~~sometimes~~ will show up

Infinities in Integration Limits

1. f con't on $[a,\infty)$ then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. f continuous on $[-\infty, b]$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

Ex:

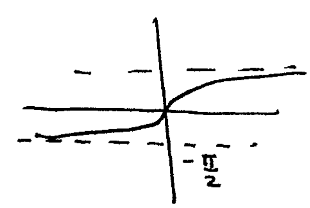
$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{b\to\infty} \int_1^b x^{-3/2}\,dx = \lim_{b\to\infty} \left.\frac{x^{-1/2}}{-1/2}\right|_1^b$$

$$= \lim_{b\to\infty} -2\left[\underbrace{\cancel{\frac{1}{\sqrt{b}}}}_{\to 0} - \frac{1}{\sqrt{1}}\right] = \boxed{2}$$

Ex:

$$\int_{-\infty}^2 \frac{2\,dx}{x^2+4} = \lim_{a\to-\infty} \int_a^2 \frac{2\,dx}{x^2+4} = \lim_{a\to-\infty} \left. 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right|_a^2$$

$$= \lim_{a\to-\infty} \left[\tan^{-1}(1) - \tan^{-1}(a)\right]$$



$$= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \boxed{\frac{3\pi}{4}}$$

Def: If the value of an Improper Integral is finite we say it converges. Otherwise, we say the integral diverges

Function ~~takes on~~ goes to infinity at integration limit(s).

3. f continuous on $(a, b]$ then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

RH Limit $(a, b]$ $\leftarrow$

4. f continuous on $[a, b)$ then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

LH limit $[a, b)$ $\rightarrow$

<u>Ex:</u>

$$\int_0^1 \frac{\theta + 1}{\sqrt{\theta^2 + 2\theta}}\,d\theta$$

infinity at 0

$$= \lim_{c \to 0^-} \int_c^1 \frac{\theta + 1}{\sqrt{\theta^2 + 2\theta}}\,d\theta$$

$u = \theta^2 + 2\theta$

$du = (2\theta + 2)\,d\theta$

$\frac{du}{2} = (\theta + 1)\,d\theta$

$$= \lim_{c \to 0^-} \frac{1}{2}\int_c^3 \frac{du}{\sqrt{u}}$$

$$= \lim_{c \to 0^-} \frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{u}\,\Big|_c^3 = \lim_{c \to 0} \sqrt{3} - \cancel{\sqrt{c}}^{\,\nearrow 0} = \boxed{\sqrt{3}}$$

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Ex: $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$ infinite at 2

$$= \lim_{c\to 2^-} \int_0^c \frac{s+1}{\sqrt{4-s^2}}\,ds$$

$$= \lim_{c\to 2^-} \left[\int_0^c \frac{s}{\sqrt{4-s^2}}\,ds + \int_0^c \frac{1}{\sqrt{4-s^2}}\,ds \right]$$

$u = 4-s^2$

$du = -2s\,ds$

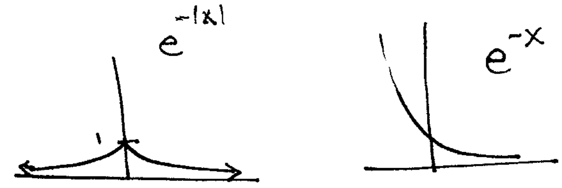
$$= \lim_{c\to 2^-} -\frac{1}{2}\int_4^{4-c^2} \frac{du}{\sqrt{u}} + \lim_{c\to 2^-} \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^c$$

$$= \lim_{c\to 2^-} -\frac{1}{\cancel{2}}\cdot \cancel{2}\sqrt{u}\,\Big|_4^{4-c^2} + \lim_{c\to 2^-} \left[\underbrace{\sin^{-1}\left(\frac{c}{2}\right)}_{\frac{\pi}{2}} - \underbrace{\sin^{-1}(0)}_{0}\right]$$

$$= \lim_{c\to 2^-} -\sqrt{4-c^2} + \sqrt{4}$$

$$0 + \sqrt{4} \quad + \quad \frac{\pi}{2} - 0$$

$$= \cancel{\boxed{\frac{\pi}{2}}} \quad \boxed{\sqrt{4} + \frac{\pi}{2}}$$



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Ex: $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$ $\qquad e^{-|x|} = \begin{cases} e^{-x} & x \geq 0 \\ e^{x} & x < 0 \end{cases}$

$= \displaystyle\int_{-\infty}^{0} e^{x} + \int_{0}^{\infty} e^{-x}$ Now use symmetry

$= 2\displaystyle\int_{-\infty}^{0} e^{x} = \lim_{b \to -\infty} 2\int_{b}^{0} e^{x}\,dx = \lim_{b \to -\infty} 2e^{x}\Big|_{b}^{0}$

$= \displaystyle\lim_{b \to -\infty} 2[e^{0} - \underbrace{e^{b}}_{0}]$ b/c $b \to -\infty$ i.e. $\frac{1}{e^{\infty}}$

$= \boxed{2}$

Infinity at an interior point on the Domain of $f(x)$.

If f becomes infinite on an interior pt. d of $[a, b]$ then

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)\,dx$$

so by prev. type, infinite at an end pt this becomes

$$= \lim_{c \to d^-} \int_a^c f(x)dx + \lim_{c \to d^+} \int_c^b f(x)\,dx$$

*** $\displaystyle\int_a^b f(x)\,dx$ Converges iff both $\displaystyle\int_a^d$ and $\displaystyle\int_d^b$ converge.

Divergent Example

Ex:

$$\int_0^1 \frac{1}{1-x}\,dx \qquad \text{infinite at } 1$$

$$= \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-} -\ln|1-x| \Big|_0^b$$

$$= \lim_{b\to 1^-} -\ln|1-b| - (\cancelto{0}{-\ln 1})$$

$$= -(-\infty) = \boxed{\infty}$$

Diverges.

Ex: Infinite at an Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} \qquad \text{Infinite at } 1$$

$$= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \qquad \text{No symmetry here...}$$

$$= \lim_{c\to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} + \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^-} \frac{3}{5}(x-1)^{5/3} \Big|_0^c + \lim_{c\to 1^+} \frac{3}{5}(x-1)^{5/3} \Big|_c^3$$

$$= \lim_{c\to 1^-} \frac{3}{5}\left[\cancelto{0}{(c-1)^{5/3}} - (-1)^{5/3}\right] + \lim_{c\to 1^+} \frac{3}{5}\left[(2)^{5/3} - \cancelto{0}{(c-1)^{5/3}}\right]$$

$$= \boxed{\frac{3}{5} + \frac{3}{5}2\sqrt[3]{4}}$$

"<u>Theorem</u>" very important fact

$$\int_1^{\infty} \frac{dx}{x^p} \qquad \text{"p- integral"}$$

<u>3 cases</u>

$p > 1$

$$= \lim_{b \to \infty} \int_1^b \frac{dx}{x^p} \qquad p > 1$$

$$= \lim_{b \to \infty} \int_1^b x^{-p}\, dx = \lim_{b \to \infty} \left. \frac{x^{-p+1}}{p+1} \right|_1^b$$

$$= \lim_{b \to \infty} \frac{1}{p+1} \Big[\underbrace{b^{-p+1}}_{\to 0} - 1 \Big] \qquad \text{Converge}$$

$\leftarrow$ Because $p > 1 \Rightarrow -p+1 < 0$

If $p < 1$

$\leftarrow$ Because $p<1 \Rightarrow -p+1 > 0$

$$= \lim_{b \to \infty} \frac{1}{p+1} \Big[\underbrace{b^{-p+1}}_{\to \infty} - 1 \Big]$$

If $p = 1$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{x}\, dx = \lim_{b \to \infty} \ln|x| \Big|_1^b$$

$$= \lim_{b \to \infty} \underbrace{\ln b}_{\to \infty} - \underbrace{\ln 1}_{\to 0} \qquad \text{Diverges}$$