7.6 Part II

Recall...

Def: if $\int_a^b f(x)dx$ is an improper integral we say the integral converges when it evaluates to a finite #. otherwise, the integral diverges.

When can't eval, can sometimes still ~~say~~ determine conv. or div. by comparing it w/ an integral that we know conv. or div.

Direct Comparison Test (DCT)

suppose f, g continuous on $[a,\infty)$,

1. if $0 \le f \le g$ and $\int_a^\infty g(x)\,dx$ conv. then $\int f$ conv. as well.

2. if $0 \le g \le f$ and $\int g$ div. then $\int f$ div. as well.

* Only ~~works~~ applies to ~~positive~~ Nonnegative functions.

Limit Comparison Test (LCT)

if f, g positive & continuous on $[a, \infty)$, ~~[illegible]~~ and

$$\lim_{x\to\infty} \frac{f}{g} = L \quad \text{where } 0 < L < \infty \quad \text{then}$$

$\int_a^\infty f$ & $\int_a^\infty g$ either both conv. or both div.

* Choosing $g(x)$ is always the hardest part!

What $g(x)$'s can we use?

$\int_a^\infty \frac{1}{x^p}\,dx$ Most common choice

Conv. $p > 1$

div. $p \le 1$.

$p \ne 1$ then

Know $= \lim_{b\to\infty} \int_a^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{p+1}\right|_a^b$

$$= \lim_{b\to\infty} \frac{1}{p+1}\left[b^{-p+1} - a^{-p+1} \right]$$

(a^{-p+1} ↑ #)

(b^{-p+1} ↑:)

$\to 0$ for $-p+1 < 0 \Rightarrow p > 1$

$\to \infty$ for $-p+1 > 0 \Rightarrow p < 1$

$p = 1$ then $\lim_{b \to \infty} \int_a^b \frac{1}{x}\, dx = \lim_{b \to \infty} \ln|x| \Big|_a^b$

$$= \lim_{b \to \infty} \left[\ln b - \ln|a| \right] \Rightarrow \text{Div.}$$

($\ln b \to \infty$; $\ln|a|$ is a #)

"p- integrals"

$\frac{1}{e^x}$ conv. $\frac{1}{\ln}$ div $-1 \le \frac{\sin\theta}{\cos\theta} \le 1$ ~~[illegible]~~

Ex:

$$\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$$ Improper at $t = 0$

on $[0, \pi]$ $\Rightarrow$ $0 \le \sin t \le 1$

$$\Rightarrow \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$$

Know $\int_0^{\pi} \frac{1}{\sqrt{t}}\, dt$ converges

$$\Rightarrow \int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$$ conv. by DCT

Ex: $\displaystyle\int_1^\infty \frac{1}{1+x^3}\,dx$ improper b/c ∞ in int. limit

on $[1,\infty)$ have $0 \le \dfrac{1}{1+x^3} \le \dfrac{1}{x^3}$

Know $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ conv. – p integral, $p = 3 > 1$

$\therefore \displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ conv as well by DCT

Ex: $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^2-1}}$ Improper at ∞ endpt. behaves like $\dfrac{1}{x}$ as $x \to \infty$

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2-1}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \cancel{\lim_{x\to\infty}\sqrt{\frac{x^2}{x^2-1}}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^2-1}}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$$

So do same thing.

Know $\displaystyle\int_2^\infty \frac{1}{x}$ Div $\Rightarrow \displaystyle\int_2^\infty \frac{1}{\sqrt{x^2-1}}$ Div. as well

by Limit Comp. Test (LCT).

Ex: $\int_0^2 \frac{dx}{1-x^2}$ Improper at $x=1$

$$= \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$\uparrow$

$$\int_0^1 \frac{dx}{1-x^2} = \lim_{b\to 1^-} \int_0^b \frac{dx}{1-x^2} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(1-x)(1+x)}$$

Partial Fractions

$$\frac{A}{1-x} + \frac{B}{1+x} \Rightarrow B(1-x) + A(1+x) = 1$$

at $x=1 \Rightarrow 2A=1 \Rightarrow A=\frac{1}{2}$

at $x=-1 \Rightarrow 2B=1 \Rightarrow B=\frac{1}{2}$

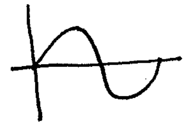
$$\Rightarrow \lim_{b\to 1^-} \left[\frac{1}{2}\int_0^b \frac{1}{1-x} + \frac{1}{2}\int_0^b \frac{1}{1+x} \right]$$

$$= \lim_{b\to 1^-} \frac{1}{2}\left[-\ln|1-x| + \ln|1+x| \right]\Big|_0^b$$

$$= \lim_{b\to 1^-} \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right|\Big|_0^b$$

$$= \lim_{b\to 1^-} \frac{1}{2}\left[\ln\frac{1+b}{1-b} - \ln|1| \right] = \infty$$ (with $\ln|1| \to 0$) so $\int_0^1 \frac{dx}{1-x^2}$ Div

$\Rightarrow \int_0^2 \frac{1}{1-x^2}$ div. as well.



Ex: $\int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{\pi-\theta}}$ Improper at $\theta = \pi$

sub: $x = \pi - \theta$ $\quad \theta = \pi - x$

$dx = -d\theta$

* $\sin(\pi - x)$
$= -\sin(x-\pi)$
↑ flip & ↑ π shift
$\Rightarrow \sin x$

$$-\int_{\pi}^{0} \frac{\sin(\pi-x)}{\sqrt{x}}\, dx = \int_0^{\pi} \frac{\sin x}{\sqrt{x}}\, dx$$

on $[0, \pi]$

$0 \le \sin x \le 1$

$\Rightarrow$ $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$

$\int_0^{\pi} \frac{1}{\sqrt{x}}\, dx$ converges $\Rightarrow$ $\int_0^{\pi} \frac{\sin x}{\sqrt{x}}\, dx$ conv.

by DCT.