Ch. 8 Sequences and Series

1360 Lecture Wednesday 10-7-09

8.1

Sequences – informal... "Lists" of numbers, get list from a rule or equation, ordered.

<u>Def:</u> An infinite sequence (sequence) of numbers is a function whose Domain is integers greater than or equal to some integer n_0.
↑start

* When we write out a sequence, we refer to the n^{th} number as the n^{th} term.

<u>Examples</u>

<u>Rule</u>	<u>sequence</u> (start at $n=1$)
1. $a_n = \sqrt{n}$	$1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots$
2. $a_n = (-1)^n$	$-1, 1, -1, 1, -1, 1, \ldots$
3. $a_n = \frac{n-1}{n}$	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
4. $a_n = 5$	$5, 5, 5, 5, 5, \ldots$

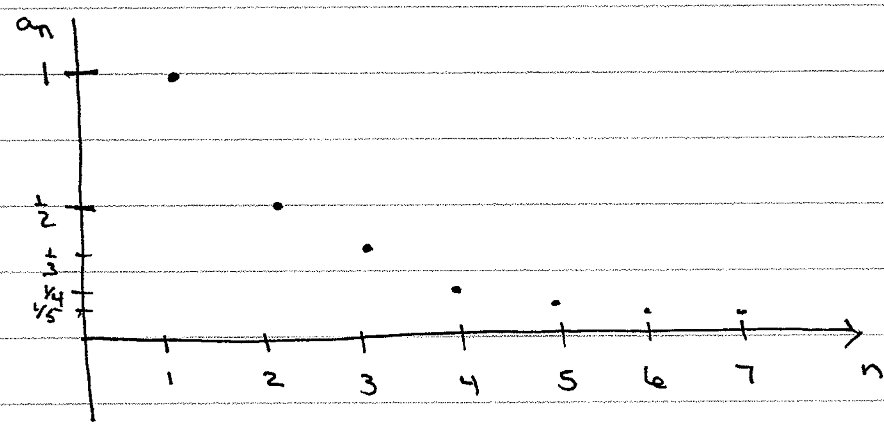
Notation: $\{a_n\}$ the sequence whose n^{th} term is given by a_n.

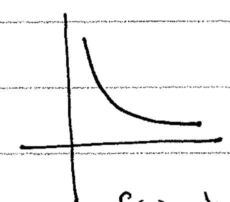
Want to take Limits of Sequences.

* Works ~~just~~ <u>similar</u> to functions

<u>Ex:</u>

$a_n = \frac{1}{n}$





$f(x) = \frac{1}{x}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

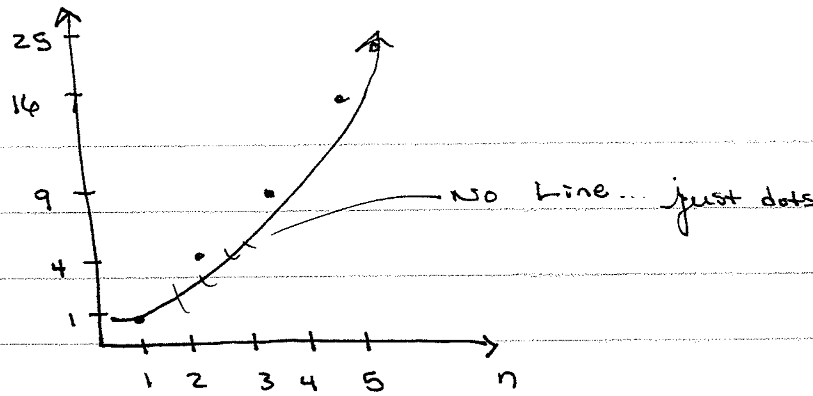
Now just Dots.

So do $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$

~~As with functions, if sequence~~

sequence converges to 0

Ex: $a_n = n^2$



$\lim_{n\to\infty} n^2 = \infty$ Seq. Diverges

Ex: $a_n = \dfrac{1}{n!}$ start at $n=1$

$\underset{n=1}{\dfrac{1}{1}}, \underset{n=2}{\dfrac{1}{2\cdot1}}, \underset{n=3}{\dfrac{1}{3\cdot2\cdot1}}, \underset{n=4}{\dfrac{1}{4\cdot3\cdot2\cdot1}}, \underset{n=5}{\dfrac{1}{5\cdot4\cdot3\cdot2\cdot1}}$

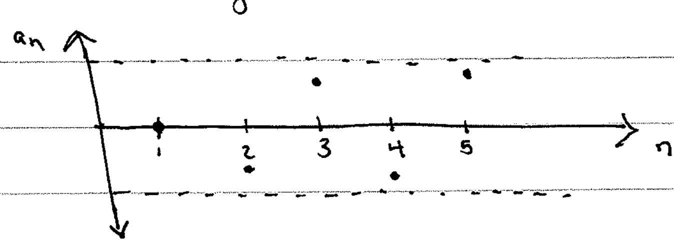
$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

$\lim_{n\to\infty} \dfrac{1}{n!} = 0$ Sequence Converges

Ex: $(-1)^{n+1} \dfrac{(n-1)}{n}$

$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$

this Diverges... Graph



half the terms a limiting to one, the other half to -1.

Overall, not limiting to a single # $\rightarrow$ Div.

Prev. Ex: Alternating Sequence.

Ex: $(-1)^{n+1}\frac{1}{n}$ $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

is $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$ alt. seq. converges.

Formally...

Def:

the sequence $\{a_n\}$ converges ~~if~~ to the number L if, $\forall \varepsilon > 0$, $\exists$ corresp. int. N s.t.

$\forall n > N \implies |a_n - L| < \varepsilon$.

If no such # L exists, the sequence Diverges.

~~*~~ Not used much in practice. Rather... other theorems tools

Def: Subsequence: select term of a given sequence to form new sequence, w/ same order but ~~poss~~ "New" restrictive rule.

Ex: $\{a_n\} = \{n\}_{n=1}^{\infty}$ seq. of positive integers

Subseq: Even ints $\{2n\}_{n=1}^{\infty}$

Odd Ints $\{2n-1\}_{n=1}^{\infty}$

Primes $\{\text{primes}\}$.

If seq. $\{a_n\}$ conv. to L then EVERY subseq. conv. to L.

1. $\therefore$ can use to prove div. if find two diff. subseq that conv. to diff. #'s
2. $\therefore$ can sometimes use to find L of conv. seq.

<u>Def:</u> Suppose seqnce $\{a_n\}$ has property

$$a_n \le a_{n+1} \qquad \forall n.$$

we say $\{a_n\}$ is a Nondecreasing Sequence.

<u>Ex:</u>

$$\left\{\frac{n}{n+1}\right\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$$

check $\quad \overset{a_n}{\frac{n}{n+1}} \overset{?}{\le} \overset{a_{n+1}}{\frac{(n+1)}{(n+1+1)}}$

$$\frac{n}{n+1} \overset{?}{\le} \frac{n+1}{n+2}$$

$$n^2+2n \overset{?}{\le} n^2+2n+1 \quad \checkmark$$

$\therefore \ a_n \le a_{n+1} \quad \forall n \Rightarrow$ Nondec.

Def: Seq. $\{a_n\}$ is <u>Bounded from Above</u> if

$\exists \; M \;$ s.t. $\; \forall n \quad a_n \leq M$

Then # M is an <u>upper bound</u> for $\{a_n\}$

If M is an u.b. and no # less than M is. u.b. then M is the <u>least upper bound</u>.

Ex: $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ has L.U.B. $M=1$.

Theorem: A nondecreasing sequence converges iff it is Bounded from above.
If a non dec. ~~con~~ seq. converges, it converges to its L.U.B.

Ex: $a_n = n - \frac{1}{n}$ conv?

as $n \to \infty$, $n \to \infty$ and $\frac{1}{n} \to 0$

so $\lim\limits_{n\to\infty} n - \frac{1}{n} = \infty$. Unbounded $\Rightarrow$ Div.

Ex: $a_n = 1 - \frac{1}{n}$ as $n \to \infty$, $\frac{1}{n} \to 0$ so $\lim\limits_{n\to\infty} 1 - \frac{1}{n} = 1$.

or could ~~also~~ say.. is nondecreasing $f(x) = 1 - \frac{1}{x}$

$f' = \frac{1}{x^2} \geq 0$ inc $\Rightarrow$ nondec

and bounded above by 1. $\Rightarrow$ conv.

Ex: Given terms, find formula for a_n.

$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

$$\left\{ \frac{(-1)^{n+1}}{n^2} \right\}_{n=1}^{\infty}$$

Ex: $0, 3, 8, 15, 24$

$1, 2, 3, 4, 5$

$$\left\{ n^2 - 1 \right\}_{n=1}^{\infty}$$

Ex: $2, 6, 10, 14, 18, \ldots$

$1, 2, 3, 4, 5$

$$\left\{ 4n - 2 \right\}_{n=1}^{\infty}$$