1360
Lecture
Friday
10-9-09

8.2 Calculating Limits of Sequences

* Works "basically" like functions.

<u>Rules</u>

$\{a_n\}$, $\{b_n\}$ two sequences of real numbers

$\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, k constant

1. $\lim_{n\to\infty} (a_n + b_n) = A + B$ sum rule
2. $\lim_{n\to\infty} (a_n - b_n) = A - B$ diff.
3. $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$ prod.
4. $\lim_{n\to\infty} (k \cdot a_n) = k \cdot A$ const. mult.
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ $B \neq 0$ quotient

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Sandwich Theorem

$\{a_n\}$, $\{b_n\}$, $\{c_n\}$ seq. of real numbers.

If $a_n \leq b_n \leq c_n$ $\forall n$ beyond some index N
~~the~~ and if

$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$ as well.

<u>Ex:</u> $\{a_n\} = \left\{ \frac{\cos n}{n} \right\}$

$-1 \leq \{\cos n\} \leq 1 \Rightarrow \frac{-1}{n} \leq \left\{\frac{\cos n}{n}\right\} \leq \left\{\frac{1}{n}\right\}$ and

$\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$ $\therefore \lim_{n\to\infty} \frac{\cos n}{n} = 0$.

Fact: Shorten the work if $|b_n| \le c_n$ and $\lim_{n\to 0} c_n = 0$ then $\lim_{n\to\infty} b_n = 0$ as well.

Ex: ~~$a_n = \frac{1}{2^n}$ then since~~

$a_n = (-1)^n \frac{1}{n}$ then certainly

$$\left|(-1)^n \frac{1}{n}\right| \le \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

then $\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$ as well.

That was fun...

Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a seq. of real #'s. If $a_n \to L$ and $f(x)$ is a function that is continuous at L and defined for all a_n, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Ex: $a_n = \sqrt{\frac{n+1}{n}}$ let $f(x) = \sqrt{x}$

look at inside... $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{n}{n} + \frac{1}{n} = 1$

so $\sqrt{1} = 1 \Rightarrow \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = 1$

Using L'Hopital's Rule !?

Suppose $f(x)$ is function defined $\forall\, x \geq n_0$ and $\{a_n\}$ is sequence s.t. $a_n = f(n)$ $\forall\, n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$

Ex: $a_n = \frac{\ln n}{n}$

$$\lim_{x\to\infty} f(x) = \frac{\ln x}{x} = \frac{\infty}{\infty} \quad \text{indet. form.}$$

$$\text{L'H} \quad = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Ex: $a_n = \left(1 + \frac{b}{n}\right)^n$

$$\lim_{x\to\infty} \left(1+\frac{b}{x}\right)^x = L$$

$L =$

$$\ln L = \ln\left(\lim_{x\to\infty} \left(1+\frac{b}{x}\right)^x\right)$$

$$= \lim_{x\to\infty} \ln\left(1+\frac{b}{x}\right)^x$$

$$= \lim_{x\to\infty} x\ln\left(1+\frac{b}{x}\right) = \infty\cdot 0 \quad \text{indet form}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{b}{x}\right)}{\frac{1}{x}}$$

* Ignore Picard's Method.

L'Hop.

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{b}{x}} \cdot \frac{-b}{x^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{b}{1+\frac{b}{x}} = b = \ln L$$

$\therefore\ L = e^b.$

Ex: $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

subseq: $b_n = (-1)^{2n}\left(1+\frac{1}{2n}\right)$ even n's

$= 1 + \frac{1}{2n}$

$\lim_{n\to\infty} b_n \quad 1 + \frac{1}{2n} = 1$

Subseq: $c_n = (-1)^{2n-1}\left(1 + \frac{1}{2n-1}\right)$ odd n's

$= -\left(1 + \frac{1}{2n-1}\right)$

$\lim c_n = -1$

since $\{b_n\}$ & $\{c_n\}$ conv. to Different limits, original sequence $\{a_n\}$ Diverges.

change to -4 on top.

#11, 18, 22, 26, 28, n!'s etc.

~~Ex:~~ Table: 8.1 p. 625

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1$ $(x > 0)$

4. $\lim_{n\to\infty} x^n = 0$ $|x| < 1$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

* ln's grow slower than linear things.
* n! Grows faster than exponentials

* $\sqrt[n]{n}$ becomes useful later on in series land

* x's in table can be replaced with any x allowed (see restrictions $x>0$, $|x|<1$)