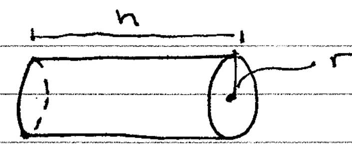
5.2 Volumes by Slicing

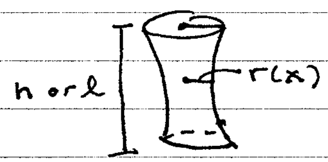
Ex:



$V = \underbrace{\pi r^2}\, h$ ← ht or length: add up all cross-section areas.

Area of infinitesimally thin cross section

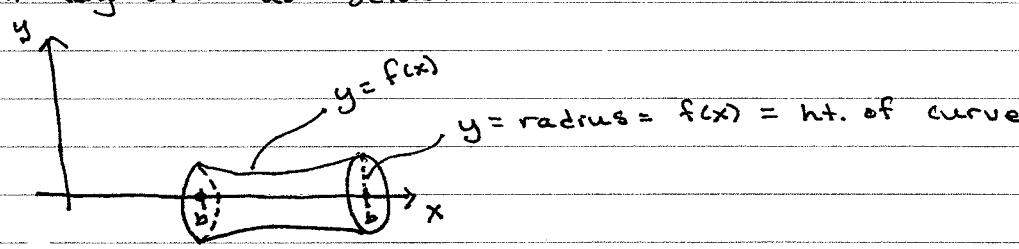
Extend this idea to cyllinder when radius changes along the length of the "cyllinder"



Suppose we know a function for radius in terms of x.
Then Area of cross section is $\pi[r(x)]^2$

* Add these areas up to get Volume.
* Adding an infinite # of infinitesimally small quantities $\Rightarrow \int_a^b$

Position solid on ~~xy~~ axes as below:



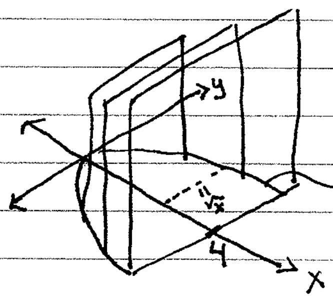
length (height) = b - a

Find area of cross section at X, A(x)
• is ~~a~~ infinetisemally thin, Add all cross-section areas to get Volume.

So do $\int_a^b A(x)\,dx$ where $A(x) = \pi[r(x)]^2$

Can Extend to General Shape... Just need an integrable function for the area of a cross section.

#2b **Ex:** Suppose a Solid lies btwn $x=0$ and $x=4$. Cross-Sections are squares w/ base in xy-plane, $\perp$ to x-axis & base runs from $y=-\sqrt{x}$ to $y=\sqrt{x}$



Formula for $A(x)$ typical cross section.

$$A(x) = (2\sqrt{x})^2$$

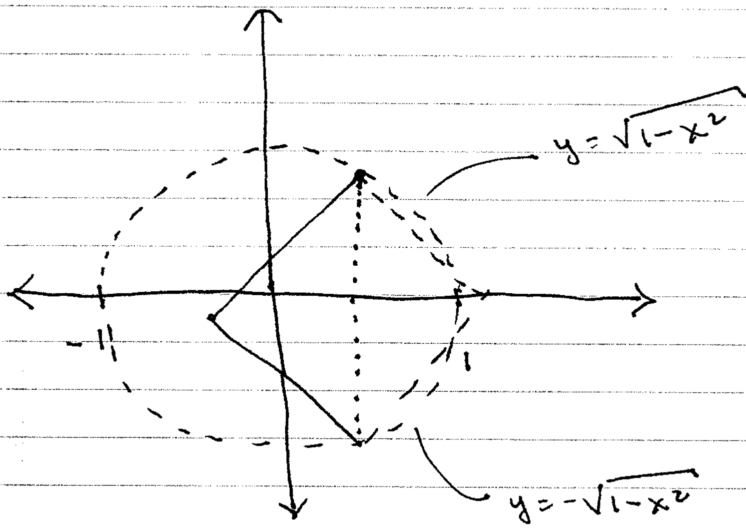
so $A(x) = 4x$

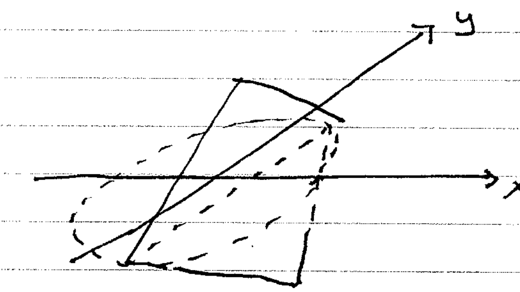
$$\int_0^4 4x\,dx = 2x^2\Big|_0^4 = \boxed{32}$$

or do #4 p. 378

#6

Ex: Solid lies btwn planes $\perp$ to x-axis at $x=-1$ and $x=1$. Cross sections $\perp$ to ~~these~~ the x-axis & btwn thes planes whose diag's run from $y=-\sqrt{1-x^2}$ to $y=\sqrt{1-x^2}$





Need area of cross section.

Know $d=2\sqrt{1-x^2}$ and side length s relates to d by $s^2+s^2=d^2$ or $2s^2=d^2$

$s^2=\frac{d^2}{2}$

Then Area $= s^2$

Sub: $A = \frac{d^2}{2}$ want i.t.o. x

sub: $A = \frac{(2\sqrt{1-x^2})^2}{2}$

Simplify

$A(x) = 2(1-x^2)$

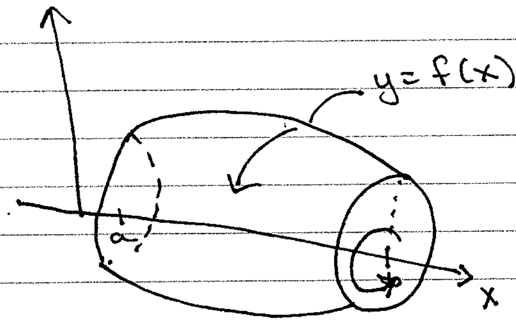
$$V(x)=\int_{-1}^{1}A(x)\,dx = 2\int_{-1}^{1}(1-x^2)\,dx = 2\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = 2\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= 2\left[\frac{2}{3}-\left(-\frac{2}{3}\right)\right] = 2\left[\frac{4}{3}\right] = \boxed{\frac{8}{3}}$$

5.3 Volumes of Solids of Revolution
* Disk & Washer Methods

The idea of "Slicing" to find volume extends to Disk & Washer methods.

Again, find Area of cross section, then integrate along the length of the Solid.



Given function $y = f(x)$, ~~rotate~~ on $[a, b]$

Rotate about x-axis to create a "Solid of Revolution"

Then cross section is Circle so $A(x) = \pi [R(x)]^2$

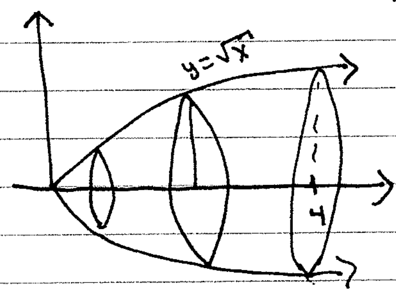
where $R(x)$ = radius of cross section.

Then

$$V = \int_a^b \pi [R(x)]^2 \, dx$$

In the above picture, $R(x) = f(x)$.

Ex: Standard 1st example...



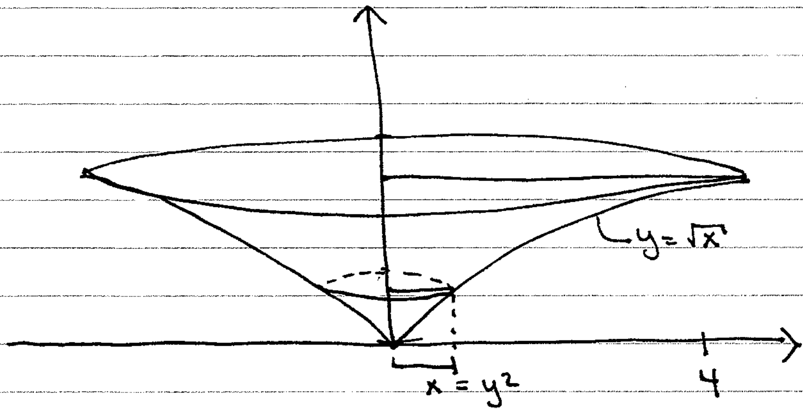
$y = \sqrt{x}$ on $[0, 4]$

Rotate about x-axis

$R(x) = \sqrt{x}$

$$V = \int_0^4 \pi (\sqrt{x})^2 \, dx = \int_0^4 \pi x \, dx = \left. \frac{\pi x^2}{2} \right|_0^4 = \boxed{8\pi}$$

Ex: Same curve... Rotate about y-axis



cross-sections ⊥ to y-axis now so integrate w.r.t. y.

since x goes from 0 to 4, y goes from 0 to 2

Radius is $x = y^2$

$$\int_0^2 \pi (y^2)^2 \, dy = \frac{\pi y^5}{5}\Big|_0^2 = \frac{\pi}{5} \cdot 32 = \boxed{\frac{32\pi}{5}}$$

Washer Method: when there's a hole in the middle of the disk...

General Formula:

Examples Tomorrow

$$\int_a^b \pi \left[R(x)^2 - r(x)^2\right] dx \qquad \text{Rotate about x-axis}$$

or

$$\int_c^d \pi \left[R(y)^2 - r(y)^2\right] dy \qquad \text{Rotate about y-axis}$$

R = Outter Radius
r = inner radius.

$$\pi R^2 - \pi r^2 = \pi\left[R^2 - r^2\right]$$

common mistake $\cancel{\pi (R-r)^2}$