5.5 Lengths of Plane Curves
5.6 Surface Area for Surfaces of Revolution

5.5

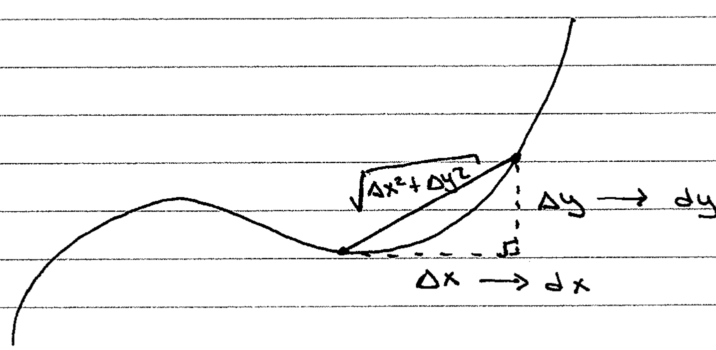
Def: A function with continuous first derivative is called "smooth.

Intuitive: No corners or cusps in the graph.

Def: If $f(x)$ is smooth on $[a,b]$ then the length of the curve $y=f(x)$ from a to b is

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Easy...
Where does this formula come from? Pythagorean Theorem!



but $f' = \frac{\Delta y}{\Delta x} \Rightarrow \Delta y = f'\cdot\Delta x$

$\Rightarrow dy = f'\cdot dx$

Substitute

$$\sqrt{\Delta x^2+\Delta y^2} = \sqrt{\Delta x^2 + (f')^2(\Delta x)^2}$$
$$= \sqrt{\Delta x^2\left[1+f'^2\right]}$$
$$= \Delta x\sqrt{1+f'^2}$$

as $\Delta x \to 0$ becomes dx in Riemann Sum.

Can do int. w.r.t. y when have $x = g(y)$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

<u>Ex:</u> $y = \sqrt{1-x^2}$, $-\frac{1}{2} \le x \le \frac{1}{2}$

$$y' = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$$

$$y' = \frac{-x}{\sqrt{1-x^2}}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx$$

$$= \int_{-1/2}^{1/2} \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$$

Set up, but do not integrate.

Suppose, $f(x)$ not smooth everywhere... how do we deal w/ discontinuities in $\frac{dy}{dx}$ $\left(\text{or } \frac{dx}{dy}\right)$?

Work with each smooth piece separately & add up the resulting lengths.

* Often times, there is a bit of Algebra before can Integrate.

Ex: #13 $x = \left(\frac{y^4}{4}\right) + \left(\frac{1}{8y^2}\right)$ $1 \le y \le 2$

$$\frac{dx}{dy} = \frac{4y^3}{4} + \frac{-2}{8y^3}$$

$$\frac{dx}{dy} = y^3 - \frac{1}{4y^3}$$

$$\int_1^2 \sqrt{1 + \left(y^3 - \frac{1}{4y^3}\right)^2}\ dy$$

$$= \int_1^2 \sqrt{1 + y^6 - \frac{1}{2} + \frac{1}{16y^6}}\ dy$$

$$= \int_1^2 \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}}\ dy$$

$$= \int_1^2 \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2}\ dy$$

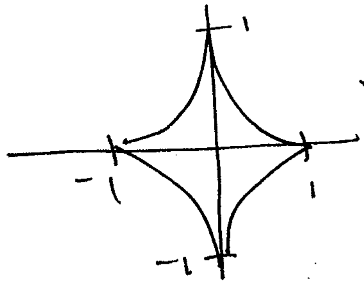
$$= \int_1^2 y^3 + \frac{1}{4}y^{-3}\ dy$$

$$= \frac{y^4}{4} + \frac{y^{-2}}{-8}\Bigg|_1^2$$

$$= \left(4 - \frac{1}{32}\right) - \left(\frac{1}{4} - \frac{1}{8}\right)$$

$=$

Ex: # 22 Get Started...



$x^{2/3} + y^{2/3} = 1 \implies y = (1 - x^{2/3})^{3/2}$

$\pm$ so take $+$

i.e. upper 1/2 plane...

Question says 1/2 of 1st quad & mult by 8

Let's set up 1st quad. Would then mult by 4

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2} \cdot \left(-\frac{2}{3} x^{-1/3}\right)$$

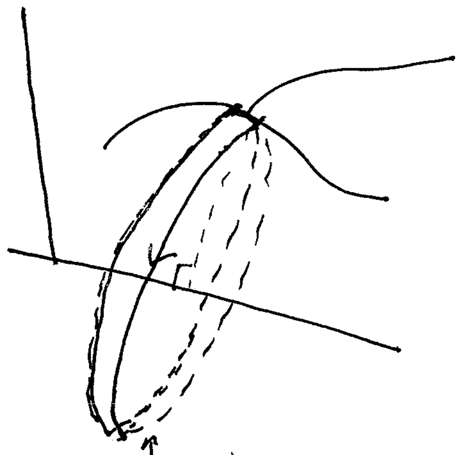
$$= \frac{-\sqrt{1 - x^{2/3}}}{x^{1/3}}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{-\sqrt{1 - x^{2/3}}}{x}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{1 - x^{2/3}}{x^2}}\, dx = \int_0^1 \sqrt{1 + \frac{1}{x^2} - \frac{x^{2/3}}{x^2}}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{1}{x^2} - \frac{1}{x^{4/3}}}\, dx$$

5.6 Surface Area - Surface of Revolution



small curve length $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

revolve about x-axis

$$C = 2\pi r = 2\pi y = 2\pi f(x)$$

revolve about x-axis

$$\therefore \quad SA = \int_a^b 2\pi f(x) \sqrt{1 + \underbrace{\left(\frac{dy}{dx}\right)}_{f'}^2}\, dx$$

revolve about y-axis i.e. $x = g(y)$

$$SA = \int_c^d 2\pi g(y) \sqrt{1 + \underbrace{\left(\frac{dx}{dy}\right)}_{g'}^2}\, dy$$

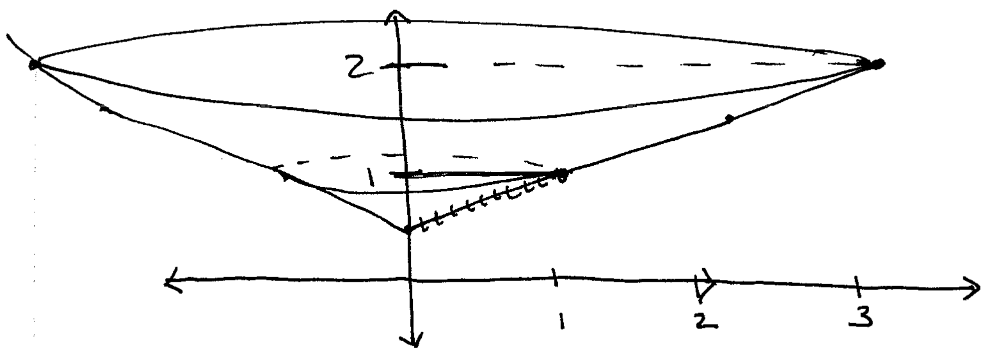
#12, #19,

#15

Ex: #12

Find SA of the cone frustrum generated by revolving the line segment $y = \left(\frac{x}{2}\right) + \frac{1}{2}$, $1 \le x \le 3$

about the y-axis



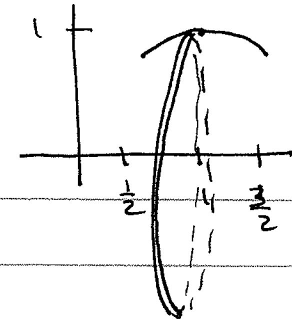
y-axis $\Rightarrow$ dy

Solve for $\quad x = 2y - 1$

Find $\frac{dx}{dy} \quad \frac{dx}{dy} = 2$

$$S = \int_1^2 2\pi \overset{x}{\overset{\downarrow}{(2y-1)}} \sqrt{1 + 4}\, dy$$

$$= 2\pi\sqrt{5} \int_1^2 2y - 1 \, dy = \boxed{4\pi\sqrt{5}}$$



Ex: #15

$y = \sqrt{2x - x^2}$ $\qquad \frac{1}{2} \le x \le \frac{3}{2}$ $\qquad$ x-axis

$\Rightarrow dx$

$$\int_{\frac{1}{2}}^{\frac{3}{2}} 2\pi \overset{\overset{y}{\downarrow}}{\sqrt{2x-x^2}} \sqrt{1 + \left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2}(2x-x^2)^{-1/2}(-2x+2)$$

$$= \frac{1-x}{\sqrt{2x-x^2}}$$

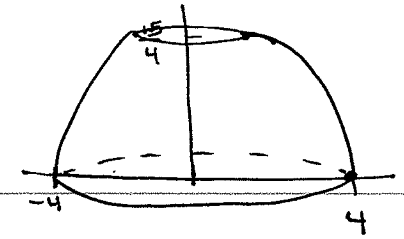
$$= 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{2x-x^2} \sqrt{1 + \frac{(1+x)^2}{2x-x^2}}\, dx$$

$$= 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{2x-x^2} \sqrt{\frac{2x-x^2}{2x-x^2} + \frac{x^2-2x+1}{2x-x^2}}$$

$$= 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{2x-x^2} \sqrt{\frac{2x - x^2 + x^2 - 2x + 1}{2x-x^2}}$$

$$= 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{2x-x^2} \cdot \frac{\sqrt{\cancel{\quad} + 1}}{\sqrt{2x-x^2}}\, dx$$

$$= 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} \cancel{\quad}\, dx = 2\pi x \Big|_{\frac{1}{2}}^{\frac{3}{2}} = 2\pi\left[\frac{3}{2} - \frac{1}{2}\right] = \boxed{2\pi}$$



dome w/ hole in top.

#19

$$x = 2\sqrt{4-y}$$

$$0 \le y \le \frac{15}{4}$$

y-axis

$$\frac{dx}{dy} = \frac{-1}{\sqrt{4-y}}$$

$$\int_0^{\frac{15}{4}} 2\pi\left(2\sqrt{4-y}\right)\sqrt{1+\left(\frac{-1}{\sqrt{4-y}}\right)^2}\,dy$$

$$= 4\pi\int_0^{\frac{15}{4}} \sqrt{4-y}\sqrt{\frac{4-y \quad + 1}{4-y}}\,dy$$

$$4\pi\int_0^{\frac{15}{4}} \sqrt{4-y}\,\frac{\sqrt{5-y}}{\sqrt{4-y}}$$