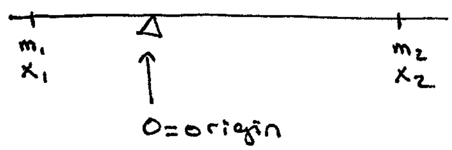
5.7 Moments & Centers of Mass

Consider bar w/ mass m_1, m_2 on either end, balanced on fulcrum. $m_1 \neq m_2$ where put fulcrum?



m_1, m_2 masses

x_1, x_2 signed dist. to origin

To know where to put fulcrum, balance out the torques

dist. to rotation pt.

$T = mgx$ So $T_1 = m_1 x_1 g$, $T_2 = m_2 x_2 g$

mass gravity

$$T_1 + T_2 = 0$$

$$m_1 x_1 g + m_2 x_2 g = 0 \qquad \text{cancel } g$$

$$\Rightarrow \sum_{i=1}^{2} m_i x_i = 0$$

If don't know x_1, x_2 Let $\bar{x}$ = c.m.

then have $\sum_{i=1}^{2} m_i (x_i - \bar{x}) = 0$ & solve for $\bar{x}$

~~Do for general~~

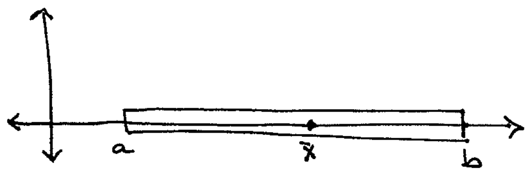
$$\sum_{i=1}^{2} m_i x_i - \sum_{i=1}^{2} m_i \bar{x} = 0$$

$$\sum_{i=1}^{2} m_i x_i - \bar{x} \sum_{i=1}^{2} m_i = 0$$

$$\Rightarrow \bar{x} = \frac{\sum_{i=1}^{2} m_i x_i}{\sum_{i=1}^{2} m_i}$$

Note: 2 not relevant when solving... do for general k.

Next... Suppose have thin rod or wire with density $\delta(x)$ as shown



$\delta(x)$ not necess. uniform along strip. But con't.

units $\frac{\text{mass}}{\text{length}}$

Now take $$\bar{x} = \frac{\sum_{i=1}^{K} x_i m_i}{\sum_{i=1}^{K} m_i}$$

Chop strip up into K ~~[illegible]~~ small pieces.
then mass of i^{th} piece is $m_i = \Delta x_i \, \delta(x_i)$

so $$\bar{x} = \frac{\sum_{i=1}^{K} x_i \, \Delta x_i \, \delta(x_i)}{\sum_{i=1}^{K} \Delta x_i m_i}$$

δ continuous, smaller & smaller pieces to get better estimates

$\boxed{\Delta x_i \to 0 \text{ or } K \to \infty \text{ equiv}}$

$$\Rightarrow \quad \bar{x} = \lim_{\Delta x_i \to 0} \frac{\sum_{i=1}^{K} x_i \, \delta(x_i) \, \Delta x_i}{\sum_{i=1}^{K} m_i \Delta x_i}$$

$\to dx$

$\int_a^b$

$$\Rightarrow \quad \bar{x} = \frac{\int_a^b x \, \delta(x) \, dx}{\int_a^b \delta(x) \, dx}$$

$$\bar{X} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx} = \frac{\overset{M_y}{M_o} = \text{Moment about origin}}{\text{Mass of System} = M}$$

Ex: Just to show it works

strip lengh $b-a$, δ = constant

$$\bar{X} \quad \frac{\int_a^b x\,\delta\,dx}{\int_a^b \delta\,dx} = \frac{\frac{\delta x^2}{2}\Big|_a^b}{\delta x\Big|_a^b} = \frac{\delta\left(\frac{b^2}{2} - \frac{a^2}{2}\right)}{\delta(b-a)}$$

$$= \frac{1}{2}\frac{(b^2-a^2)}{(b-a)} = \frac{1}{2}\frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}}$$

$$= \frac{1}{2}(b+a)$$

$= \text{midpt of } [a,b]$ average

Ex Varying Density # 9

Rod w/ density function $\delta(x) = 1 + \frac{1}{\sqrt{x}}$ positioned on $1 \le x \le 4$. Find c.m.

$$\bar{x} = \frac{M_0}{M} = \frac{\int_1^4 x\left(1 + \frac{1}{\sqrt{x}}\right)dx}{\int_1^4 \left(1 + \frac{1}{\sqrt{x}}\right)dx} = \frac{\int_1^4 (x + \sqrt{x})\,dx}{\int_1^4 1 + x^{-1/2}\;dx}$$

$$= \frac{\left.\frac{x^2}{2} + \frac{x^{3/2}}{3/2}\right|_1^4}{\left.x + \frac{x^{1/2}}{1/2}\right|_1^4} = \frac{\left(\frac{4^2}{2} + \frac{2 \cdot 4^{3/2}}{3}\right) - \left(\frac{1}{2} + \frac{2}{3}\right)}{\left(4 + 2 \cdot 4^{1/2}\right) - (1 + 2)}$$

$$= \frac{\left(8 + \frac{16}{3}\right) - \left(\frac{7}{6}\right)}{(8) - (3)} = \frac{\frac{48}{6} + \frac{32}{6} - \frac{7}{6}}{5} = \frac{\frac{73}{6}}{5} = \boxed{\frac{73}{30}}$$

Now extend this idea to 2D plate in the xy-plane.

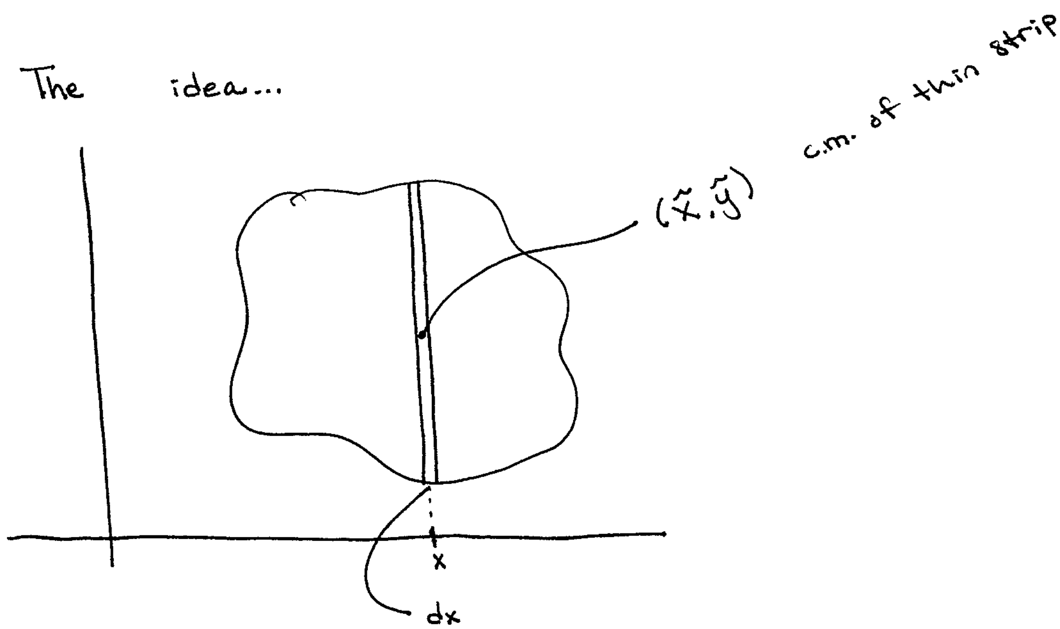
Just do $(\bar{x}, \bar{y})$ for c.m.

<u>Eqns:</u> $M_x = \int \tilde{y}\, dm$ $\qquad M_y = \int \tilde{x}\, dm$

mass $M = \int dm$ $\qquad$ c.m. $= (\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M} = \frac{\int \tilde{x}\, dm}{\int dm} \qquad \bar{y} = \frac{M_x}{M} = \frac{\int \tilde{y}\, dm}{\int dm}$$

~~[illegible]~~ The idea...



assume: so thin, const density $\Rightarrow$ strip c.m. $(x, \tfrac{1}{2}\text{-way})$

$(\tilde{x}, \cancel{\tfrac{1}{2}\text{-way}}\ \tilde{y})$

Then for whole plate need dm = mass

$dm = \delta \cdot dA = \delta \cdot \underbrace{l \cdot w}_{\text{of strip}}$

<u>Steps</u> draw a typical strip

Vertical

$l =$

$w = dx$

$\tilde{x} = x$

area $= dA = l \cdot w$

mass $= dm = \delta\, dA$

$\tilde{y} =$ half-way

$M_x = \int \tilde{y}\, dm$ $\qquad M_y = \int \tilde{x}\, dm$ $\qquad M = \int dm$

Horiz. Strip

$l =$

$w = dy$

$\tilde{x} = \frac{1}{2}$-way

area $= dA = l \cdot w$

mass $= dm = \delta\, dA$

$\tilde{y} = y$

$M_x = \int \tilde{y}\, dm$ $\qquad M_y = \int \tilde{x}\, dm$ $\qquad M = \int dm$

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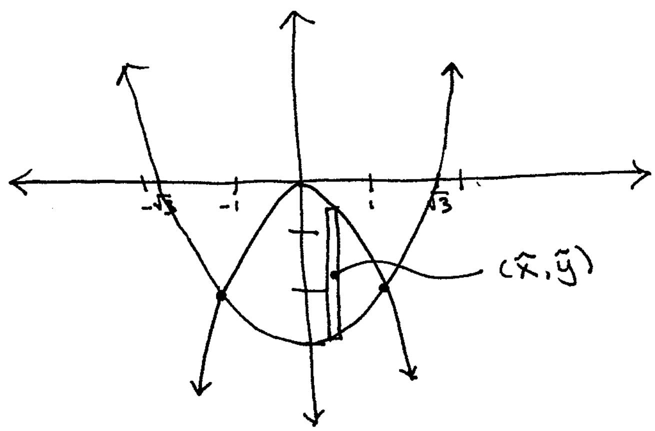
Constant Density

Ex: Region enclosed by $y = x^2 - 3$ and $y = -2x^2$



$x^2 - 3 = -2x^2$

$3x^2 - 3 = 0$

$x = \pm 1$

* Int. in terms of x easy $\Rightarrow dx$
$\Rightarrow$ Vertical Strip

$l = (-2x^2) - (x^2 - 3) = -3x^2 + 3$

$w = dx$ area $= dA = lw = (-3x^2+3)\,dx$

$\tilde{x} = x$ mass $= dm = \delta\, dm = \delta(-3x^2+3)\,dx$

$\tilde{y} = \dfrac{(x^2-3) + (-2x^2)}{2} = \dfrac{-x^2-3}{2}$

midpt = Avg = $\frac{\text{add}}{2}$

$$\bar{x} = \frac{M_y}{M} = \frac{\int_{-1}^{1} \tilde{x}\, dm}{\int_{-1}^{1} dm} = \frac{\int_{-1}^{1} x\delta(-3x^2+3)\,dx}{\int_{-1}^{1} \delta(-3x^2+3)\,dx} =$$

$$\bar{y} = \frac{M_x}{M} = \frac{\int_{-1}^{1} \tilde{y}\, dm}{\int_{-1}^{1} dm} = \frac{\int_{-1}^{1} \frac{-x^2-3}{2} \cdot \delta(-3x^2+3)\,dx}{\int_{-1}^{1} \delta(-3x^2+3)\,dx} =$$