APPM 1360 FALL 2009 REVIEW #2 SOLNS.

1(a) $$\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt \underset{u=\sqrt{t}}{=} 2\int e^u\,du = 2e^u + c = \underline{\underline{2e^{\sqrt{t}}+c}}$$

(b) $$\int \frac{\tan(\theta)}{2\sec\theta+1}\,d\theta = \int \frac{\sin\theta\,d\theta}{2+\cos\theta} \underset{u=2+\cos(\theta)}{=} \underline{\underline{-\ln|2+\cos(\theta)|+C}}$$

$$\frac{\tan\theta}{2\sec(\theta)+1}\cdot\frac{\cos(\theta)}{\cos(\theta)} = \frac{\sin(\theta)}{2+\cos\theta}$$

(c) $$\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx \underset{\text{IBP}}{=} x^2\sin^{-1}(x^2) + \sqrt{1-x^4}\,\Big|_0^{1/\sqrt{2}} = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-1/4} - 1$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \sqrt{3/4} - 1 = \underline{\underline{\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1}}$$

$u = \sin^{-1}(x^2)$ $\quad dv = 2x\,dx$

$du = \dfrac{2x}{\sqrt{1-x^4}}$ $\quad v = x^2$

$$\int 2x\sin^{-1}(x^2)\,dx = x^2\sin^{-1}(x^2) - \underbrace{\int \frac{2x^3}{\sqrt{1-x^4}}\,dx}_{\substack{u=1-x^4 \\ du=-4x^3\,dx}} = x^2\sin^{-1}(x^2) + \sqrt{1-x^4} + C$$

$$\frac{-1}{2}\int\frac{du}{\sqrt{u}} = -\sqrt{u} + C$$

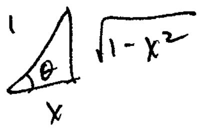
(d) $$\int \frac{y\,dy}{y^2-2y-3} = \int\frac{y\,dy}{(y+1)(y-3)} = \frac{3}{4}\int\frac{1}{y-3}\,dy + \frac{1}{4}\int\frac{1}{y+1}\,dy = \underline{\underline{\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1| + C}}$$

Partial Fractions: $\dfrac{y}{(y+1)(y-3)} = \dfrac{A}{y-3} + \dfrac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3)$

$y=-1 \rightarrow -1 = -4B \rightarrow B = 1/4$

$y=3 \rightarrow 3 = 4A \rightarrow A = 3/4$

1(e). $$\int \frac{(1-x^2)^{1/2}}{x^4}\,dx = \int \frac{\sin(\theta)}{\cos^4(\theta)}\cdot -\sin(\theta)\,d\theta = \int -\frac{\sin^2(\theta)}{\cos^4(\theta)}\,d\theta$$



$\sin(\theta) = \sqrt{1-x^2}$

$\cos(\theta) = x$

$-\sin(\theta)\,d\theta = dx$

$$= -\int \tan^2(\theta)\sec^2(\theta)\,d\theta$$

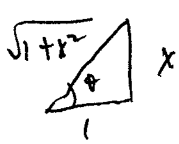
$u = \tan(\theta),\ du = \sec^2(\theta)$

$$= -\int u^2\,du$$

$$= -\frac{u^3}{3} + C$$

$$= -\frac{1}{3}\tan^3(\theta) + C = \underline{\underline{\frac{-1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C}}$$

(f) $$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2(\theta)\,d\theta}{\tan^2(\theta)\sec(\theta)} = \int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$



$\sec(\theta) = \sqrt{1+x^2}$

$\tan(\theta) = x$

$\sec^2(\theta)\,d\theta = dx$

$$= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\cdot\frac{1}{\cos(\theta)}\,d\theta$$

$$= \int \frac{\cos(\theta)\,d\theta}{\sin^2(\theta)} = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$$

$u = \sin(\theta)$

$$= \frac{-1}{\sin(\theta)} + C$$

$$= \underline{\underline{-\frac{\sqrt{1+x^2}}{x} + C}}$$

oops!

2. $$\int \frac{1}{x - \sqrt{x}}\,dx = \cancel{\int \frac{dx}{\sqrt{x}(1-\sqrt{x})}} = \int \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx$$

oops!

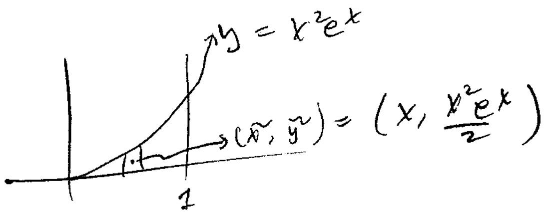
$$= \int \frac{2\,du}{u} = 2\ln|u| + C = \underline{\underline{2\ln|\sqrt{x}-1| + C}}$$

$u = \sqrt{x} - 1$

$du = \frac{1}{2\sqrt{x}}\,dx$

doh!

3.



Note, $M = \text{density} \cdot \text{area} = \int_0^1 \delta \cdot x^2 e^x \, dx = \int dm \underset{\text{IBP}}{=} \delta (e-2)$,

$$\int_0^1 x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x \Big|_0^1 = e - 2$$

$$\int x^2 e^x \, dx = \underbrace{x^2 e^x - \int 2x e^x \, dx}_{\substack{u = x^2,\ dv = e^x \\ du = 2x\,dx,\ v = e^x}} = \underbrace{x^2 e^x - \left[2x e^x - \int 2e^x\right]}_{\substack{u = 2x,\ dv = e^x \\ du = 2,\ v = e^x}}$$

and, $M_x = \int \tilde{y} \, dm = \frac{\delta}{2} \int_0^1 x^4 e^{2x} \underset{\text{IBP}}{=} \delta \frac{(e^2 - 3)}{8}$,

and, $M_y = \int \tilde{x} \, dm = \delta \int_0^1 x^3 e^x \, dx \underset{\text{IBP}}{=} \delta (6 - 2e)$, see §7.2, #37(a)

so, $\boxed{\bar{x} = \frac{M_y}{M} = \frac{6-2e}{e-2}, \quad \bar{y} = \frac{M_x}{M} = \frac{e^2-3}{8(e-2)}.}$

4. $(x+1) \frac{dy}{dx} = y^2 + 1, \quad y(0) = \pi/4$

$$\frac{1}{y^2+1} \frac{dy}{dx} = \frac{1}{x+1} \Rightarrow \int \frac{1}{y^2+1} \, dy = \int \frac{1}{x+1} \, dx \Rightarrow \tan^{-1}(y) = \ln|x+1| + C$$

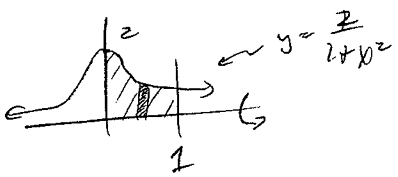
$$y = \tan(\ln|x+1| + C)$$

$$\frac{\pi}{4} = y(0) = \tan(C)$$

so $C = \tan^{-1}(\pi/4)$,

and $\underline{\underline{y = \tan(\ln|x+1| + \tan^{-1}(\pi/4))}}$.

5.



Disk: $\Delta V = \pi r^2 \Delta x$, $r = \frac{2}{1+x^2}$ $\Rightarrow$ $V = \int_0^1 \pi \left(\frac{2}{1+x^2}\right)^2 dx = 4\pi \int_0^1 \frac{1}{(1+x^2)^2} dx$

Note $\int \frac{1}{(1+x^2)^2} dx = \int \frac{\sec^2(\theta)\, d\theta}{\sec^4(\theta)} = \int \cos^2(\theta)\, d\theta$

$\sqrt{1+x^2}$, x, 1, θ

$\sec(\theta) = \sqrt{1+x^2}$

$\tan(\theta) = x$

$\sec^2(\theta)\, d\theta = dx$

$$= \int \frac{1+\cos(2\theta)}{2} d\theta$$

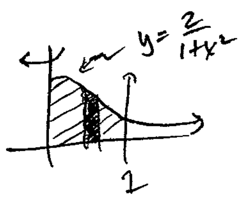
$$= \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]$$

$$= \frac{1}{2}\left[\theta + \frac{2\sin(\theta)\cos(\theta)}{2}\right] + C$$

$$= \frac{1}{2}\left[\tan^{-1}(x) + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right] + C$$

So, $4\pi \int_0^1 \frac{1}{(1+x^2)^2} dx = 2\pi \left[\tan^{-1}(x) + \frac{x}{1+x^2}\right]\Big|_0^1 = \underline{2\pi\left(\frac{\pi}{4} + \frac{1}{2}\right)}$

6.



Shell: $\Delta V = 2\pi r h\, \Delta x$, $r = x$, $h = \frac{2}{1+x^2}$ $\Rightarrow$ So, $V = \int_0^1 2\pi \cdot x \cdot \frac{2}{1+x^2} dx = 2\pi \int_0^1 \frac{2x}{1+x^2} dx$

$u = 1+x^2$, $du = 2x\,dx$

$$= 2\pi \ln|1+x^2| \Big|_0^1$$

$$= \underline{2\pi \ln(2)}$$

7. (a) $\int_0^1 -\ln(x)\,dx = \lim_{a\to 0^+} \int_a^1 -\ln(x)\,dx = \lim_{a\to 0^+} -x(\ln(x)-1)\Big|_a^1$

$= \lim_{a\to 0^+} 1 + a\ln(a) - a$

$\int \ln(x)\,dx = x\ln(x) - \int dx = x\ln(x) - x + C = x(\ln(x)-1)+C$

$u = \ln(x),\ dv = dx$

$du = \frac{1}{x}dx,\ v = x$

Note, $\lim_{a\to 0^+} a\ln(a) \overset{\text{"}0\cdot-\infty\text{"}}{=} \lim_{a\to 0^+} \frac{\ln(a)}{1/a} \overset{\text{"}-\infty/\infty\text{"}}{\underset{\text{LH}}{=}} \lim_{a\to 0^+} \frac{1/a}{-1/a^2} = \lim_{a\to 0^+} -a = 0$

So $\int_0^1 -\ln(x)\,dx = \lim_{a\to 0^+} 1 + a\ln(a) - a = 1$, so integral converges.

(b) Note, $0 \le \int_\pi^\infty \frac{1+\sin(x)}{x^2}\,dx \le \int_\pi^\infty \frac{2}{x^2}\,dx$

since $-1 \le \sin(x) \le 1 \Rightarrow 0 \le 1+\sin(x) \le 2$

and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim_{b\to\infty} -2x^{-1}\Big|_\pi^b = \lim_{b\to\infty} \frac{-2}{b} + \frac{2}{\pi} = 2/\pi$ converges

so $\int_\pi^\infty \frac{1+\sin(x)}{x^2}\,dx$ converges by direct comparison w/ $\int_\pi^\infty \frac{2}{x^2}\,dx$.

7 (c) $\displaystyle\int_1^\infty \frac{1}{e^x - x - 1}\,dx$, compare w/ $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx$,

note $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty} \left.\frac{-1}{e^x}\right|_1^b = \lim_{b\to\infty} \frac{-1}{e^b} + \frac{1}{e} = 1/e$, ← converges

and (using Limit Comparison Test:) $\cancel{\displaystyle\lim_{x\to\infty} \frac{a_x}{b_x}}$ $\displaystyle\lim_{x\to\infty} \frac{1}{e^x - x - 1} \cdot \frac{e^x}{1} = \lim_{x\to\infty} \frac{e^x}{e^x - x - 1}$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{e^x - 1} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1$$

Since $0 < 1 < \infty$ and since $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx$ converges

we have that $\displaystyle\int_1^\infty \frac{1}{e^x - x - 1}\,dx$ converges by Limit Comparison Test.

8 (a) $\displaystyle\lim_{n\to\infty} \left(1 - \frac{3}{n^2}\right)^n \overset{\text{"}1^\infty\text{"}}{=} \lim_{n\to\infty} e^{\frac{\ln(1 - 3/n^2)}{1/n}}$ "0/0"

$$\overset{L'H}{=} \lim_{n\to\infty} e^{\frac{6/n^3}{1 - 3/n^2} \cdot \frac{-n^2}{1}} = \lim_{n\to\infty} e^{\frac{-6}{n(1 - 3/n^2)}} = e^0 = 1$$

so $a_n \to 1$ as $n \to \infty$

(converges to 1)

(b) Note $\log_b(a) = \dfrac{\ln(a)}{\ln(b)}$, so $a_n = \dfrac{\log_n e}{\log_n n} = \dfrac{\ln(e)}{\ln(n)} = \dfrac{1}{\ln(n)}$

and $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$, so $a_n \to 0$ as $n \to \infty$

(converges to 0)

8(c) Note $-1 \le (-1)^n \le 1 \Rightarrow -n \le n(-1)^n \le n$ for $n>0$

So $1-n \le 1+n(-1)^n \le 1+n$ for $n>0$

So $\frac{1-n}{n!} \le \frac{1+n(-1)^n}{n!} \le \frac{1+n}{n!}$ for $n>0$

now $\lim\limits_{n\to\infty} \frac{1-n}{n!} = \lim\limits_{n\to\infty} \frac{1}{n!} - \frac{1}{(n-1)!} = 0$

$\lim\limits_{n\to\infty} \frac{1+n}{n!} = \lim\limits_{n\to\infty} \frac{1}{n!} + \frac{1}{(n-1)!} = 0$

So $\lim\limits_{n\to\infty} \frac{1+n(-1)^n}{n!} = 0$ by Squeeze Theorem

So sequence converges to 0.

(d) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n \frac{n+1}{n}$ = oscillates between 1 and -1,

So sequence diverges.

(e) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^n e^{-2n} = \lim\limits_{n\to\infty} \left(\frac{n}{e^2}\right)^n = +\infty$

so sequence diverges.

(f) $\lim\limits_{n\to\infty} \frac{1+\ln(5n^2)}{3n} = \lim\limits_{n\to\infty} \frac{1+\ln(5)+2\ln(n)}{3n} \overset{\text{"}\infty/\infty\text{"}}{\underset{LH}{=}} \lim\limits_{n\to\infty} \frac{2/n}{3} = 0$

So sequence converges to 0.

9(a) $\sum_{n=2}^{\infty} \frac{100}{n^{3/2}} = 100 \cdot \sum_{n=2} \frac{1}{n^{3/2}}$ a convergent p-series

so $\sum_{n=2}^{\infty} \frac{100}{n^{3/2}}$ is convergent since it is a constant multiple of a convergent series.

(b) $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^{1/n}$

Note $\lim_{n\to\infty} 3^{1/n} = 1$, and $\lim_{n\to\infty} n^{1/n} = 1$

so $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n\to\infty} 3^{1/n}}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1 \neq 0$

so series <u>diverges</u> by Div. Test.

(c) $\sum_{n=3}^{\infty} \frac{2}{(n-1)(n-2)} = \sum_{n=3}^{\infty} \frac{2}{n-2} - \frac{2}{n-1} = \left(2 - \cancel{\frac{2}{2}}\right) + \left(\cancel{\frac{2}{2}} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \cdots\right.$

by partial fractions

$= 2 - \lim_{n\to\infty} \frac{2}{n-1} = 2$

$\frac{2}{(n-1)(n-2)} = \frac{A}{n-2} + \frac{B}{n-1}$

so series is a convergent telescoping series converging to 2.

9(d) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1}}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)\, 2^{n-1}}{3^{n-1}\cdot 3^2} = \underbrace{\sum_{n=1}^{\infty} \frac{-1}{9}\left(\frac{-2}{3}\right)^{n-1}}_{\text{Geo series}}$

Note $(-1)^n = (-1)^{n-1}(-1)$

$3^{n+1} = 3^n \cdot 3 = 3^{n-1}\cdot 3^2$

Geo series w/ $a = -1/9$ and $r = -2/3 \Rightarrow |r| < 1$

so, $\sum_{n=1}^{\infty} \frac{-1}{9}\left(\frac{-2}{3}\right)^{n-1} = \frac{-1/9}{1 + 2/3} = -\frac{1}{9}\cdot\frac{3}{5} = \frac{-1}{15}$

Geo. series converges to $-1/15$

(e) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^2-1}}$

Note $\lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}\sqrt{1-1/n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-1/n^2}} = 1 \neq 0$

so series diverges by Div. Test.

(f) $\sum_{n=2}^{\infty}\left(\frac{1}{\ln\pi}\right)^n = \frac{1}{(\ln\pi)^2} + \frac{1}{(\ln\pi)^2}\cdot\frac{1}{\ln\pi} + \frac{1}{(\ln\pi)^2}\cdot\left(\frac{1}{\ln\pi}\right)^2 + \cdots = \underbrace{\sum_{n=0}^{\infty}\frac{1}{(\ln\pi)^2}\left(\frac{1}{\ln\pi}\right)^n}_{\text{Geo series.}}$

Note $e < \pi \Rightarrow \ln(e) < \ln(\pi) \Rightarrow 1 < \ln(\pi) \Rightarrow \frac{1}{\ln(\pi)} < 1 \Rightarrow$ converges

so $\sum_{n=2}^{\infty}\left(\frac{1}{\ln\pi}\right)^n = \frac{1/(\ln\pi)^2}{1 - 1/\ln\pi} = \frac{1}{(\ln\pi)^2}\cdot\frac{\ln(\pi)}{\ln(\pi)-1} = \frac{1}{\ln(\pi)(\ln(\pi)-1)}$

Geo series converges to

10. $\lim_{n\to\infty} n^{1/n} \overset{\text{"}\infty^0\text{"}}{=} \lim_{n\to\infty} e^{\frac{\ln(n)}{n}} \overset{L'H}{=} \lim_{n\to\infty} e^{1/n} = e^0 = 1$

11. $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \overset{\text{"}1^\infty\text{"}}{=} \lim_{n\to\infty} e^{\frac{\ln(1+x/n)}{1/n}}$ "$\frac{0}{0}$"

$\overset{L'H}{=} \lim_{n\to\infty} e^{\frac{-x/n^2}{1+x/n}\cdot\frac{-n^2}{1}} = \lim_{n\to\infty} e^{\frac{x}{1+x/n}} = e^x$

$\lim_{n\to\infty} \frac{x}{n} = 0.$