APPM 1360 Spring 1996

FINAL EXAM

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your recitation time and recitation instructor, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books and class notes are NOT permitted. A calculator and a one-page crib sheet are allowed. Please start each new problem on a new page.

1. (24 points) Find dy/dx for each of the following functions. (Be sure that you actually calculate dy/dx!)

   (a) $y = (\log_{10} x + x^{-10})^{10}$

   (b) $y = (2 + \sin x + \cos x)^x$

   (c) $x(t) = t - \sin t$ and $y(t) = 1 - \cos t$

   (d) $r = \ln(\theta)$ for $\theta > 0$

2. (32 points) Evaluate the following integrals analytically. If the integral is definite, first state whether the integral coverges or diverges, and then if the integral converges, find its value. Show all work clearly.

   (a) $\displaystyle\int_1^{\sqrt{e}} \frac{2}{x\sqrt{1-4(\ln x)^2}}\,dx$

   (b) $\displaystyle\int x(\ln x)^2\,dx$

   (c) $\displaystyle\int \frac{x^2}{x^2-9}\,dx$

   (d) $\displaystyle\int_0^{\infty} \frac{1}{(4+x^2)^{3/2}}\,dx$

3. (20 points) The intensity $L(x)$ of light at a depth x feet below the surface of the ocean satisfies $\dfrac{dL}{dx} = -kL$ with the initial condition $L(0) = L_0$. As a diver, you know from experience that diving to 18 feet cuts the intensity of light in half. You cannot work without artificial light when intensity falls below $\frac{1}{10}$ of the surface intensity, L_0. How deep can the diver work without artificial light?

4. (20 points) Find the interval of convergence for the following series. Clearly specify if the series diverge, converge absolutely, or converge conditionally at the endpoints.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (\ln n)(x-2)^n$

5. (35 points) Consider the integral $I(x) = \int_2^x \ln t \, dt$.

   (a) Derive the Taylor series for $\ln t$ with center $a = 2$.
   (b) Integrate the above series to arrive at a Taylor series representation, centered at $a = 2$, for $I(x) = \int_2^x \ln t \, dt$.
   (c) Determine how many terms of the series in part (b) are needed to numerically evaluate $I(2.1)$ with an error less than 10^{-3}.
   (d) Estimate $I(2.1)$ by numerically evaluating the predicted number of terms in the series.
   (e) Analytically evaluate the integral to find the exact value of $I(2.1)$. (Do not use the series representation or numerical integration on your calculator for this part.)
   (f) Compare the true value of $I(2.1)$ from part (e) with the estimated value of $I(2.1)$ from part (d) to show that the difference is indeed less than 10^{-3}.
   (g) In three sentences or less, discuss how the problem changes if we consider the integral $I(1.9)$. Be as specific as possible for a 7:30-10:30 AM final exam.

6. (20 points) Each August, the Persius meteor shower occurs when the earth passes through the debris trail left by Halley's comet. This trail has been forming for approximately 150,000 years and produces one of the best meteor showers in any given year. The orbit of Halley's comet is an ellipse with the sun at one focus. The semi-minor axis of the comet orbit is 4.56 astronomical units and the eccentricity is approximately $e = 0.97$. (One astronomical unit, denoted AU, is about 92,600,000 miles.)

   (a) Assume that the major axis of the ellipse is centered on the x-axis. Calculate the appropriate parameters (in units of AUs) and then write the standard form of the equation for the ellipse.
   (b) Calculate the minimum and maximum distance of Halley's comet from the sun. Report these distances in terms of AUs.

7. (28 points) Consider the two curves, $r_1 = 3 \sin \theta$ and $r_2 = 2 - \sin \theta$.

   (a) Carefully and neatly graph the two equations on the same coordinate system. Clearly label the curves, r_1 and r_2. Also, clearly label the x and y intercepts.
   (b) Find all points of intersection of the two curves and clearly identify them on your graph.
   (c) Find the area of the region inside r_1 and outside r_2.
   (d) Set up the integral to calculate the arc length of the portion of curve r_2 that is inside curve r_1.
   (e) Numerically evaluate the arc length in part (d). Show what key strokes you used on your calculator.

8. (21 points) Consider the parametric equations $x(t) = \sin t$ and $y(t) = (\tan t)(1 + \sin t)$.

   (a) Neatly and carefully graph the equations for $0 \le t \le 2\pi$, $t \ne \pi/2$. (This curve is called a strophoid.)
   (b) Find the slopes of the tangent lines to the curve at the origin.
   (c) Eliminate the parameter and then show that the line $x = 1$ is a vertical asymptote. (Hint: You might start with the reference triangle corresponding to $x = \sin t$, although there are other ways to work this problem.)