APPM 1360 **Final Exam** May 11, 1998

On the front of your bluebook, please write:

your name,
your instructor's name, and
the color of your exam.
Also, make a grading list for 10 questions.

This exam is worth 200 points. There are 10 questions, each worth 25 points. **You may skip any 2 questions**. On your grading list, mark zeroes next to the numbers of the 2 questions you want to skip. [*If no numbers are marked, the first 8 questions will be graded.*] Answer all parts of the remaining 8 questions; you are not allowed to skip parts of several questions. Show your work in your bluebook. | Box | in your answers. Calculators are permitted; if you use the calculator in an essential way, state how you used it.

1. a) Find the area of the region bounded by the parabola $y = x - x^2$ and the line $y = -x$.

b) Find the coordinates of the center of mass of a thin plate of constant density _ covering this region.

2. Consider the sphere obtained by rotating the curve $x = \sqrt{a^2 - y^2}$ about the y-axis.

a) The sphere is filled with liquid up to a height (h) above the bottom of the sphere, as shown. ($0 \le h \le 2a$) Find the volume of the liquid.

b) [*To check your answer:*] The sphere is half full (or half empty) when h = a. What is the volume of liquid in this special case?

3. a) Determine whether each of the following integrals converges or diverges. Justify each answer.

(i) $\displaystyle\int_0^\infty \frac{dx}{(x-2)^2}$, (ii) $\displaystyle\int_1^\infty \frac{e^{-x}}{x^2}\,dx$, (iii) $\displaystyle\int_0^1 \frac{e^{-x}}{x^2}\,dx$.

b) Calculate the following limit, if it exists:

$$\lim_{x \to 0^+} \left[x \int_x^1 \frac{e^{-t}}{t^2}\,dt\right].$$

4. a) Determine whether each of the following sequences converges or diverges as $n \to \infty$. Justify your answer. If a sequence converges, then find its limiting value.

$$c_n = \left(1 - \frac{2}{n^2}\right)^n, \qquad d_n = \left(1 - \frac{3}{n}\right)^{n^2}, \qquad e_n = \left(1 - \frac{4}{n^2}\right)^{n^2}.$$

b) Find a simplified expression for S_N, the N^{th} partial sum of the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right).$$

c) Does the series in (b) converge? If so, to what? If not, why not?

5. Every infintely repeating decimal represents a rational number, p / q, where p and q are integers with no common factors. The number π (=3.141592653589793...) is known to be irrational, so its decimal expansion never repeats. It can be approximated by various rational numbers, including $\frac{22}{7}$ (=3.142857 142857 142857 ...).

a) Find the rational number represented by 3.14 14 14 14 14 ...

b) Find the rational number represented by 3.1416 1416 1416 ...

In each case, make sure that the numerator and denominator of your rational number have no common factors.

c) Evaluate the following series, if the sums exist, and state which one is larger:

$$\sum_{n=0}^{\infty} \frac{3^n + 4^n}{7^n}, \quad \sum_{n=0}^{\infty} \frac{(3+4)^n}{7^n}.$$

6. Let $f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n \; n^2}$ be a function defined by its power series.

a) Find the radius of convergence of the series.

b) Find the interval of convergence of the series. (Check the endpoints.)

c) What is the power series representation of $\frac{df}{dx}(x)$?

d) What is the radius of convergence of the power series for $\frac{df}{dx}(x)$?

e) What is the interval of convergence of the power series for $\frac{df}{dx}(x)$? (Check the endpoints.)

7. a) What is the Maclaurin series for $e^{-x^2/2}$?

b) Find the Maclaurin series for $\int_0^x e^{-t^2/2}\,dt$?

c) For what range of x does the series in (b) converge? Justify your answer.

d) How many terms in this series are needed to evaluate $\int_0^1 e^{-t^2/2}\,dt$ with an error less than 10^{-3}?

e) Find an approximate value for $\int_0^1 e^{-t^2/2}\,dt$.

8. A curve in the x-y plane is defined parametrically by

$$x(t) = t + t^3, \quad y(t) = 1 + t^2.$$

a) Calculate $\frac{dy}{dx}$ as a function of t.

b) Calculate $\frac{d^2y}{dx^2}$ as a function of t.

c) Observe that the curve passes through the point {x=2, y=2}. Find the equation of the line tangent to the curve at (2, 2).
[*Warning: this answer should involve no t's.*]

d) Find the quadratic approximation (*i.e.*, the Taylor polynomial of order 2) of y(x) about x = 0.

9. A hyperbola is defined by the equation:

$$x^2 - \frac{y^2}{4} + 1 = 0 .$$

a) Sketch the graph of this curve.

b) Give a parameterization of each branch of the curve in the form:
x = X(t), y = Y(t). Be sure to include the domain of each function X(t), Y(t).

c) For each branch of the hyperbola, find the point (or points) on that branch closest to the point (0, 10).

10. a) Sketch the curve defined by the equation (in polar coordinates)

$$r = 2(1 - \cos \theta) .$$

[*If you use your calculator for part (a), please state on your exam that you have done so.*]

b) Where does this curve intersect the circle: r = 2?

c) Find the area of the region shared by the two curves (*i.e.*, inside both curves).

d) Find the total length of the curve in (a).

e) Is the length of the curve in (a) larger or smaller than the circumference of the circle in (b)?

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a grading table, with two questions marked that you want to skip?