1. (20 points) Evaluate the following integrals. Be sure to show all your work.

   (a) Integration by parts with $u = \ln(2x)$ and $dv = xdx$
   $\int x \ln(2x)\, dx = \frac{x^2}{2}\ln(2x) - \int \frac{x}{2}\, dx = \frac{x^2}{2}\ln(2x) - \frac{x^2}{4} + C$

   (b) Complete the square: $\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C$
   or using $u = \sqrt{x}$ will get $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{x}\,\sqrt{4-x}} = \int \frac{2\,du}{\sqrt{4-u^2}} = 2\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + C$

   (c) Divergent improper integral: $\int_0^4 \frac{dx}{(x-2)^4} = \lim_{b\to 2^-} \int_0^b \frac{dx}{(x-2)^4} + \lim_{d\to 2^+} \int_d^4 \frac{dx}{(x-2)^4}$
   $= \lim_{b\to 2^-}\left[\frac{-1}{3(x-2)^3}\right]_0^b + \lim_{d\to 2^+}\left[\frac{-1}{3(x-2)^3}\right]_d^4 = \left(\infty - \frac{1}{24}\right) + \left(\frac{1}{24} + \infty\right)$

2. (15 points) Determine whether the following integrals converge or not. Explain your reasoning. Be sure to fully support your answer.

   (a) $\int_1^\infty \frac{4}{\sqrt{x^2+5}}\, dx$ diverges by limit comparison test with $\int_1^\infty \frac{4}{x}\, dx$

   (b) Diverges. U-subsitiution $u = \ln x$ leads to $\int_3^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty} \ln(\ln x))\Big|_3^b = \infty$

3. (15 points)

   (a) $\frac{-4x+3}{(x^2+1)(x-2)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-2)}$ where $A = 1$, $B = -2$ and $C = -1$.

   (b) Evaluating $\int \frac{-4x+3}{(x^2+1)(x-2)}\, dx = \int \frac{x}{(x^2+1)}\, dx + \int \frac{-2}{(x^2+1)}\, dx - \int \frac{1}{(x-2)}\, dx$
   $= \frac{1}{2}\ln(x^2+1) - 2\tan^{-1} x - \ln|x-2| + C$

4. (30 points) For each sequence $\{a_n\}$ below, determine if the sequence converges, and if so, to what limit. If the sequence diverges, state this. Be sure to fully support your answer.

   (a) Since $\frac{-1}{n^2} \le \frac{\sin n}{n^2} \le \frac{1}{n^2}$ then $a_n = \frac{\sin n}{n^2}$ by the Sandwich Theorem.

   (b) Using L'Hopital's rule three times shows that $\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{\ln^3(3)\, 3^n}{6}$ diverges.

   (c) $\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = e^{-2}$ since $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$ for any x.

   (d) Let $L = \lim_{n\to\infty} n^{(2/\ln n)}$. Then $\ln L = \lim_{n\to\infty} \frac{2}{\ln n}\ln n = 2$. Hence $\lim_{n\to\infty} n^{(2/\ln n)} = e^2$.

5. (20 points) Determine if each series below converges or diverges. If possible, for each convergent series determine the sum of the series. Be sure to fully support your answer.

   (a) $\sum_{n=0}^\infty \left(1 - \frac{2}{n}\right)^n$ diverges by the n-th term test since $\lim_{n\to\infty}\left(1 - \frac{2}{n}\right)^n = e^{-2}$

   (b) $\sum_{n=3}^\infty \frac{1}{n\ln n}$ diverges by the integral comparison test with the divergent $\int_3^\infty \frac{dx}{x\ln x} = \infty$ from problem 2b. Since $f(x) = \frac{1}{x\ln x}$ is continuous, positive and decreasing for $x \ge 3$, it satisfies the three conditions required to apply the integral test.

   (c) The sum of two convergent geometric series:
   $\sum_{n=0}^\infty \frac{2^n+1}{3^{2n}} = \sum_{n=0}^\infty \frac{2^n}{9^n} + \sum_{n=0}^\infty \frac{1}{9^n} = \frac{1}{1-2/9} + \frac{1}{1-1/9} = \frac{9}{7} + \frac{9}{8} = \frac{135}{56}$