1. (20 points) Determine whether the following series converge absolutely, conditionally, or diverge. Clearly justify your answer.

   (a) $\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3n}$ diverges by LCT with harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ since $\lim_{n\to\infty} \left(\frac{n^2+2}{n^3+3n}\right) \Big/ \left(\frac{1}{n}\right) = 1$.

   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges conditionally since $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges by direct comparison with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ while $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test since $a_n = \frac{\ln n}{\sqrt{n}}$ is positive, decreasing and $\lim_{n\to\infty} a_n = 0$.

   (c) $\sum_{n=1}^{\infty} \frac{3^n n^3}{n!}$ converges absolutely by the ratio test since
   $\lim_{n\to\infty} \frac{3^{n+1}(n+1)^3}{(n+1)!} \cdot \frac{n!}{3^n n^3} = \lim_{n\to\infty} \frac{3(n+1)^2}{n^3} = 0 = \rho < 1$.

2. (20 points) For each power series, determine the values of x for which the series converges absolutely, and converges conditionally. For each series, clearly state the radius of convergence.

   (a) $\sum_{n=0}^{\infty} \frac{2^n x^n}{(n+2)!}$ has infinite radius of convergence and converges absolutely for all finite x since by the ratio test $\lim_{n\to\infty} \left|\frac{2^{n+1}x^{n+1}}{(n+1+2)!} \cdot \frac{(n+2)!}{2^n x^n}\right| = \lim_{n\to\infty} \left|\frac{2x}{(n+3)}\right| = |2x| \lim_{n\to\infty} \frac{1}{(n+3)} = 0$.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n (2x-5)^n}{n}$ has a center of $a = \frac{5}{2}$, a radius of convergence $r = \frac{1}{2}$,converges absolutely for $2 < x < 3$ and converges conditionaly at $x = 3$. The ratio test $\lim_{n\to\infty} \left|\frac{(2x-5)^{n+1}}{n+1} \cdot \frac{n}{(2x-5)^n}\right| = |2x-5| \lim_{n\to\infty} \frac{n}{n+1} = |2x-5| = \rho < 1$ leads to $2 < x < 3$ for interval of absolute convergence. At $x = 2$ the series becomes the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n}$ which diverges. At $x = 3$ the series becomes the alternating harmonic series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which converges conditionaly.

3. (20 points) Consider the following computations.

   (a) Calculate the first four non-zero terms of the **Maclaurin** series for $f(x) = (1+x)^{2/3}$.
   This is a binomial series with $m = 2/3$, hence the first four non-zero terms
   $(1+x)^m \approx 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3$ are $(1+x)^{2/3} \approx 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3$.

   (b) Determine the (full) **Maclaurin** series for $f(x) = x^2 \cos(x^2)$.
   Since $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, then
   $x^2 \cos(x) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n)!} = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \cdots$

4. (20 points) We wish to approximate $\ln(1.1)$ by using the third-order Taylor polynomial $P_3(x)$ of $\ln(x)$ centered at $a = 1$.

   (a) Calculate $P_3(x)$. Since $f = \ln x$, then $f' = 1/x$, $f'' = -1/x^2$, $f''' = 2/x^3$ and $f''''(x) = -6/x^4$, which leads to $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$ and $f''''(1) = -6$. Hence
   $P_3(x) = f(1) + \dfrac{f'(1)(x-1)}{1!} + \dfrac{f''(1)(x-1)^2}{2!} + \dfrac{f'''(1)(x-1)^3}{3!} = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3}$

   (b) Use $P_3(x)$ to estimate $\ln(1.1)$. (You may leave your answer in terms of fractions.)
   $\ln(1.1) \approx P_3(1.1) = (0.1) - \dfrac{(0.1)^2}{2} + \dfrac{(0.1)^3}{3}$

   (c) Estimate the magnitude of the error associated with using $P_3(x)$ to approximate $\ln(1.1)$.
   From the Alternating Series Theorem, $|\text{error}| \le \left|\dfrac{f''''(1)(x-1)^4}{4!}\right|_{x=1.1} = \dfrac{(0.1)^4}{4}$.

5. (20 points) Using series, solve the initial value problem $y' + 2y = 0$ where $y(0) = 1$.
   Assume $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$, then
   $2y = 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \cdots$ and
   $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$.
   Substituting these into $y' + 2y = 0$ and collecting like powers of x leads to
   $(2a_0 + a_1) + (2a_1 + 2a_2)x + (2a_2 + 3a_3)x^2 + (2a_3 + 4a_4)x^3 + \cdots = 0$.
   Equating coefficients to 0 leads to $a_1 = -2a_0$, $a_2 = -a_1 = 2a_0$, $a_3 = -2a_2/3 = -4a_0/3$, $a_4 = -2a_3/4 = 2a_0/3$ and so on. Hence

$$\begin{aligned} y &= a_0 - 2a_0x + 2a_0x^2 - \frac{4a_0x^3}{3} + \frac{2a_0x^4}{3} + \cdots \\ &= a_0 + a_0(-2x) + \frac{a_0(-2x)^2}{2!} + \frac{a_0(-2x)^3}{3!} + \frac{a_0(-2x)^4}{4!} + \cdots \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} \\ &= a_0\, e^{-2x}. \end{aligned}$$

   From the initial condition $y(0) = 1 = a_0\, e^0 = a_0$. Hence $y = e^{-2x}$.