1. (20 points) Do the following sequences $\{a_n\}_1^\infty$ converge or diverge? If a sequence converges and it is possible to determine its value, then do so. Be sure to explain your reasoning.

   (a) $a_n = \dfrac{n^3+e^n}{2e^n+1}$ converges to 1/2 since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n \dfrac{e^n(n^3e^{-n}+1)}{e^n(2+e^{-1})} = 1/2$.

   (b) $a_1 = \dfrac{1}{4}$ and $a_{n+1} = 2a_n$ for all $n \geq 1$ diverges since a_n's increase by a factor of 2 starting from $a_1 = 1/4$. Since a_n can be written $a_n = 2^{n-2}$ for $n \geq 0$, then $\lim_{n\to\infty} 2^{n-2} = \infty$.

   (c) $a_{n+1} > a_n$ (a_n is a nondecreasing sequence) and $0 < a_n < 2$ for all $n \geq 1$ (a_n is bounded from above) converges by the Nondecreasing Sequence Theorem. You can't determine the limit of a_n, but you do know that $0 < \lim_{n\to\infty} a_n < 2$.

2. (20 points) Determine whether the following infinite series converge or diverge. Justify your answers.

   (a) $\sum_{n=1}^{\infty}\left(1+\dfrac{3}{n}\right)^n$ diverges by the n^{th} term test since $\lim_{n\to\infty} a_n = e^3$.

   (b) $\sum_{n=1}^{\infty}\dfrac{(-1)^n n}{2n^2+3n}$ converges by the Alternating Series Theorem since (1) $a_n = \dfrac{1}{2n+3} > 0$; (2) $\dfrac{1}{2(n+1)+3} \leq \dfrac{1}{2n+3}$; (3) $\lim_{n\to\infty} a_n = 0$. However, $\sum_{n=1}^{\infty}\left|\dfrac{(-1)^n n}{2n^2+3n}\right|$ diverges by the LCT with the divergent harmonic series $\sum_{n=1}^{\infty}\dfrac{1}{n}$. Hence, the series converges conditionally.

   (c) $\sum_{n=1}^{\infty}\ln\left(1+\dfrac{1}{n^2}\right)$ converges by the LCT with the convergent p-series $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$.

3. (20 points) Determine where the following power series converge absolutely, converge conditionally, and diverge.

   (a) $\sum_{n=1}^{\infty} n^n(x-4)^n$ has zero radius of convergence since the ratio test indicates that $\lim_{n\to\infty}\left|\dfrac{(n+1)^{n+1}(x-4)^{n+1}}{n^n(x-4)^n}\right| = |x-4|\lim_{n\to\infty}\dfrac{(n+1)^{n+1}}{n^n} < 1$ is not valid for any finite value of x other than $x=4$. Thus the series converges aboulutely at the point $x=4$, converges conditionally nowhere, and diverges for $|x-4|>0$.

   (b) the ratio test indicates that the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n(2x-1)^n}{n}$ converges absolutely when $\lim_{n\to\infty}\left|\dfrac{(2x-1)^{n+1}n}{(n+1)(2x-1)^n}\right| = |2x-1|\cdot 1 < 1$ which leads to $0 < x < 1$. At $x=0$ the series becomes the divergent harmonic series $\sum_{n=1}^{\infty}\dfrac{1}{n}$. At $x=1$ the series becomes the conditionally convergent alternating harminic series $\sum_{n=1}^{\infty}\dfrac{(1)^n}{n}$. Thus the series converges absolutely for $0 < x < 1$, converges conditionally at $x = 1$, diverges for $x = 0$ and $\left|x - \frac{1}{2}\right| > \frac{1}{2}$.

4. (20 points) Consider the function $f(x) = e^{2x}$.

   (a) Starting from the definition, determine the Taylor series of $f(x)$ around $x = 1$. Calculating derivatives of e^{2x} and evaluating them at $x = 1$, leads to $f^{(n)}(1) = 2^n e^2$. Hence the Taylor series near $x = 1$ becomes
   $\sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n = e^2 + 2e^2(x-1) + \frac{4e^2}{2!}(x-1)^2 + \frac{8e^2}{3!}(x-1)^3 + \cdots$.

   (b) Determine the interval of convergence for your series in part (a). The ratio test indicates that the series converges absolutely when $\lim_{n\to\infty}\left|\frac{(2)^{n+1}(x-1)^{n+1}n!}{(n+1)!2^n(x-1)^n}\right| = 2|x-1|\cdot 0 < 1$ which is true for all finite values of x. Thus the series converges absolutely on the interval $-\infty < x < \infty$.

   (c) What are the first three non–zero terms of the Taylor polynomial around $x = 1$?
   $P_2(x) = e^2 + 2e^2(x-1) + \frac{4e^2}{2!}(x-1)^2$.

   (d) Estimate of the remainder of the polynomial in part (c) when $x = 1/2$. Leave your answer in terms of fractions, factorials, etc. Evaluating the series in part (a) at $x = 1/2$ results in an alternating series. Hence if one used $P_2(1/2)$ to estimate $f(1/2)$, the error would be bounded by $|f(1/2) - P_2(1/2)| \le \frac{8e^2}{3!}(1/2)^3 = \frac{e^2}{6}$.

5. (20 points) Consider the following steps for evaluating $f(x) = \int_0^x \cos(t^2)\, dt$.

   (a) Determine the Taylor series for $\cos(t^2)$ around $t = 0$. Since the Taylor series near $t = 0$ for $\cos(t) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!} = 1 - t^2/2! + t^4/4!2 - t^6/6! + \cdots$ converges for all t, replacing t with t^2 results in the Taylor series $\cos(t^2) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{(2n)!} = 1 - t^4/2! + t^8/4! - t^{12}/6! + \cdots$.

   (b) Using your series from part (a), calculate $f(x) = \int_0^x \cos(t^2)\, dt$. Integrating the series in part (a) results in $f(x) = \int_0^x \cos(t^2)\, dt = \sum_{n=0}^{\infty}\int_0^x \frac{(-1)^n t^{4n}}{(2n)!}\, dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} =$
   $x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots$.

   (c) What is the interval of convergence of your series in part (b)? Be sure to explain your reasoning. Since the original series for $\cos(t^2)$ in part (a) converges absolutely for all t, so does the integral of the series. Hence, the series for $f(x)$ converges absolutely for $-\infty < x < \infty$.