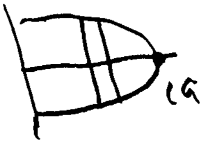
APPM 1360 Final Exam spring 2005

1. 

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 , \quad dm = 2b\sqrt{1-(x/a)^2}\,dx$$

$$\left[\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} = \frac{\int_{x=0}^{a} x\cdot[2b\sqrt{1-(x/a)^2}]\,dx}{\int_0^a 2b\sqrt{1-(x/a)^2}\,dx}\right]$$

$$\tilde{x} = x , \; \tilde{y} = 0$$

$$\left[\bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} = 0\right]$$

2. $$\frac{dy}{dx} + \frac{y}{x} = x \qquad IF = e^{\int \frac{1}{x}dx} = x \Rightarrow$$

$y(1) = 1$ $\qquad x\frac{dy}{dx} + y = x^2 \Rightarrow \frac{d}{dx}(xy) = x^2$

$$\Rightarrow xy = \frac{x^3}{3} + C , \quad y(1) = 1 \Rightarrow C = 2/3$$

$$\boxed{y = \frac{x^2}{3} + \frac{2}{3x}}$$

check: $\quad y' + \frac{y}{x} = \frac{2x}{3} - \frac{2}{3x^2} + \frac{x}{3} + \frac{2}{3x^2} = x$ ✓

3. a) $$\int x e^{kx}\,dx = \frac{1}{k}xe^{kx} - \int \frac{e^{kx}}{k}\,dx = \boxed{\frac{1}{k}xe^{kx} - \frac{1}{k^2}e^{kx} + C}$$

$k = \text{const}$, $\quad u = x \quad v = \frac{e^{kx}}{k}$

$\qquad du = dx \quad dv = e^{kx}dx$

b) $$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

$x = \sin\theta \quad dx = \cos\theta\,d\theta$
$$= \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_0^{\pi/2} = \boxed{\frac{\pi}{4}}$$

c. $\int_0^\infty \frac{dx}{(x-2)^2}$ diverges $p = 2$: finite point $x = 2$

4. $f(x) = \frac{1}{x^2(x^2+1)}$

a. $\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

$1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

$x^3: \quad 0 = A + C$
$x^2: \quad 0 = B + D$
$x: \quad 0 = A$
$x^0: \quad 1 = B$

$\Rightarrow C = 0, \quad D = -B = -1 \; : \; \frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$ ✓

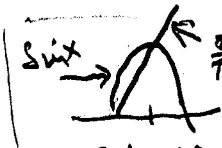
b. $\int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx = -\frac{1}{x} - \tan^{-1} x + C$

5. (a) $\int_1^\infty \frac{1}{x^3+1}\,dx$ limit compare to $\int_1^\infty \frac{dx}{x^3} \Rightarrow$ Conv $p=3$

$\lim_{x\to\infty} \frac{\frac{1}{x^3+1}}{1/x^3} = 1$

(b) $\int_1^\infty \ln\left(1+\frac{1}{x}\right) dx$ limit compare to $\int_1^\infty \frac{dx}{x} \Rightarrow$ DIV $p=1$

$\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = 1$



(c) $\int_0^{\pi/2} \frac{1}{(\sin x)^{1/2}}\,dx$ Compare $\sin x \geq \frac{2}{\pi}x \Rightarrow \frac{1}{\sin x} \leq \frac{\pi}{2x}$

$\Rightarrow \frac{1}{(\sin x)^{1/2}} \leq \left(\frac{\pi}{2x}\right)^{1/2}$

$\leq \sqrt{\frac{\pi}{2}} \int_0^{\pi/2} \frac{1}{x^{1/2}}\,dx$ conv. $p = 1/2$

6. $\{a_n\}_1^\infty$ a) $\frac{n^2}{n+1} \to \infty$ DIV b) $\cos n\pi$ Div. osc. between ± 1

c) $|a_n| \leq \tan\left(\frac{1}{n}\right)$ all $n \geq 1$

So a_n is bounded. But it can oscillate

$\Rightarrow$ a_n might converge or might diverge. Not enough info to tell

L3

7. a) $\sum_{1}^{\infty} n^2 e^{-n^3}$ use integral test $\int_1^{\infty} x^2 e^{-x^3}\,dx = \left(\frac{e^{-x^3}}{-3}\right)_1^{\infty}$ conv.

b) $\sum_{2}^{\infty} \frac{1}{\ln n}$ $n > \ln n$, $n > 2$ $\Rightarrow \frac{1}{n} < \frac{1}{\ln n} \Rightarrow \sum \frac{1}{\ln n} > \sum \frac{1}{n}$ $\Rightarrow$ Div $p=1$

direct comparison test

c) $\sum \frac{n!}{(2n)!}$ use ratio test $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2n+2)!}\,\frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} \to 0$ conv.

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8. use $\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 \dots$

$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1 - t^2 + t^4 \dots)\,dt = x - \frac{x^3}{3} + \frac{x^5}{5} \dots$

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9. $f(x) = \cos 2x$. a) $f(x) = 1 - \frac{(2x)^2}{2!}$

$|R_2| \le \frac{(2x)^4}{4!}$ $\left|R_2\left(\tfrac{1}{2}\right)\right| \le \frac{1}{4!} = \frac{1}{120}$ $\left|R_2\left(\tfrac{1}{3}\right)\right| \le .008$

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10. $\sum_{1}^{\infty} \frac{x^n}{3^n \sqrt{n}}$ interval: a) Ratio test $\Rightarrow \left|\frac{x}{3}\frac{\sqrt{n+1}}{\sqrt{n}}\right| \to \left|\frac{x}{3}\right| < 1$

$x=3 \Rightarrow \sum \frac{1}{\sqrt{n}}$ Div $p = 1/2$ $x=-3$ $\sum \frac{(-)^n}{\sqrt{n}}$ cond. conv AST

interval $\boxed{-3 \le x < 3}$

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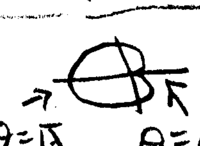
11. $r = 1 - \cos\theta$

a) length



$L = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$L = 2\int_0^{\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\,d\theta$

(use $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$) $L = 2\int_0^{\pi} \sqrt{2(1-\cos\theta)}\,d\theta = 2\int_0^{\pi}\sqrt{2\cdot 2\sin^2\theta}$,

$L = 4\int_0^{\pi} \sin\theta\,d\theta = -4(\cos\theta)_0^{\pi} = 8$ //

note $\sin\theta \ge 0$, $0 \le \theta \le \pi$

b) $A = 2\int_0^{\pi} \frac{1}{2} r^2\,d\theta = \int_0^{\pi} (1-\cos\theta)^2 = \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$ //

$= 3\pi/2$

$$Ax^2 + Bxy + Cy^2 + \dots = 0$$ L4

12. $xy = 4$ a) $A = C = 0$, $B = 1$

$$\Delta = B^2 - 4AC = 1 > 0 \quad \text{Hyperbola}$$

b) $\tan 2\alpha = \dfrac{B}{A-C} = \dfrac{1}{0} \Rightarrow 2\alpha = \pi/2 \qquad \alpha = \pi/4$

c) $x = \frac{1}{\sqrt{2}}(x' - y')$

$y = \frac{1}{\sqrt{2}}(x' + y')$

$xy = 1 \Rightarrow$

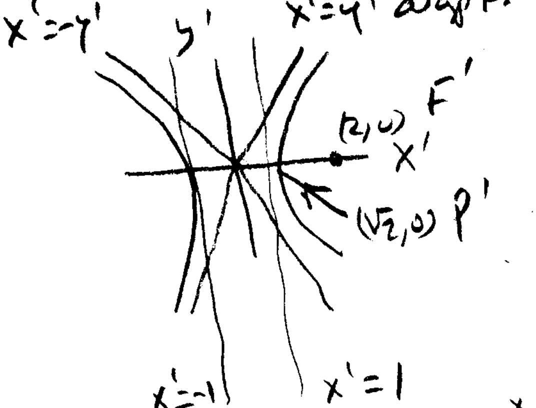
$$\left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = 1$$

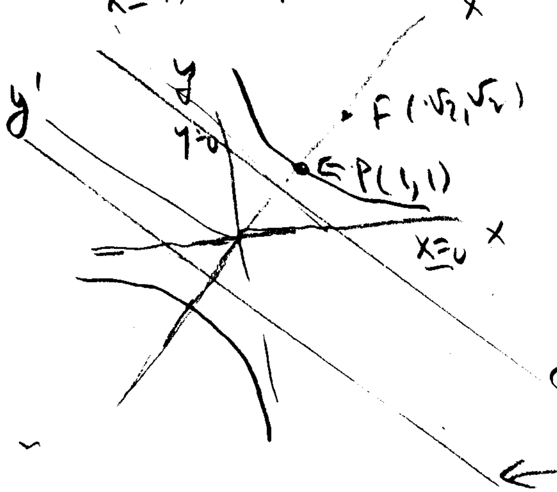
$a = b = \sqrt{2} \qquad c^2 = a^2 + b^2 = 4$

$c = 2$

$e = \frac{c}{a} = \sqrt{2}$

Directrices $x' = \pm\frac{a}{e} = \pm 1$





asymptotes $x = 0$, $y = 0$

Directrices $x' = 1$: $x = \frac{1}{\sqrt{2}}(1 - y')$

$y = \frac{1}{\sqrt{2}}(1 + y')$

$\Rightarrow x + y = \sqrt{2}$

$x' = -1 \quad x + y = -\sqrt{2}$