

APPM 1360

Exam 1

June 16, 2006

1. (40 points) Consider the region enclosed by the curves  $y = x^3$  and  $y = x^{1/3}$ .

(a) (10 points) Calculate the volume of the region rotated about the  $x$ -axis using the **washer** method.

Solutions:  $R(x) = x^{1/3}$ ,  $r(x) = x^3$  because  $(x^{1/3} \geq x^3$  for  $0 \leq x \leq 1$ )

$$\begin{aligned} V &= \int_0^1 \pi (R^2(x) - r^2(x)) dx = \int_0^1 \pi (x^{2/3} - x^6) dx \\ &= \pi \left( \int_0^1 x^{2/3} dx - \int_0^1 x^6 dx \right) = \pi \left( \frac{3}{5} - \frac{1}{7} \right) \\ &= \frac{16\pi}{35} \end{aligned}$$

(b) (10 points) Compute the same volume using the **shell** method.

Solutions:  $h(y) = y^{1/3} - y^3$  because  $x = y^{1/3} \geq x = y^3$  for  $0 \leq y \leq 1$ .

$$\begin{aligned} V &= \int_0^1 2\pi y h(y) dx = \int_0^1 2\pi y (y^{1/3} - y^3) dx \\ &= \int_0^1 2\pi (y^{4/3} - y^4) dx = 2\pi \left( \frac{3}{7} - \frac{1}{5} \right) \\ &= \frac{16\pi}{35} \end{aligned}$$

(c) (10 points) Setup, but **do not** evaluate, the integral for the volume rotated about the line  $y = 1$  using the **washer** method.

Solutions:  $R(x) = 1 - x^3$ ,  $r(x) = 1 - x^{1/3}$

$$\begin{aligned} V &= \int_0^1 \pi (R^2(x) - r^2(x)) dx \\ &= \int_0^1 \pi \left( (1 - x^3)^2 - (1 - x^{1/3})^2 \right) dx \end{aligned}$$

(d) (10 points) Setup, but **do not** evaluate, the integral for the volume rotated about the line  $x = 1$  using the **shell** method.

Solutions:

$$V = \int_0^1 2\pi(1-x) (x^{1/3} - x^3) dx$$

2. (20 points)

(a) (10 points) Find the solution to I.V.P.

$$\begin{aligned} xy' + y &= -\sin(x) \\ y\left(\frac{\pi}{2}\right) &= 0 \end{aligned}$$

Solutions: Put ODE in standard form

$$\begin{aligned} y' + p(x)y &= q(x) \\ p(x) &= \frac{1}{x} \\ q(x) &= -\frac{\sin(x)}{x} \end{aligned}$$

Then,

$$\begin{aligned} y(x) &= \frac{1}{v(x)} \int v(x)q(x)dx \\ v(x) &= \exp\left(\int p(x)dx\right) \\ &= \exp\left(\int \frac{1}{x} dx\right) \\ &= \exp(\ln x) \\ &= x \end{aligned}$$

So,

$$\begin{aligned} y(x) &= \frac{1}{x} \int x \left(-\frac{\sin(x)}{x}\right) dx \\ &= -\frac{1}{x} \int \sin(x) dx \\ &= \frac{\cos(x) + C}{x} \end{aligned}$$

Evaluate initial condition:

$$y\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right) + C}{\frac{\pi}{2}} = 0$$

Thus,  $C = 0$  and  $y(x) = \frac{\cos(x)}{x}$ .

- (b) (10 points) Verify that your solution in (a) solves the I.V.P (you can buy the correct answer for 10 points).

Solutions:

$$\begin{aligned} y &= \frac{\cos(x)}{x} \\ y' &= -\frac{x \sin(x) + \cos(x)}{x^2} \end{aligned}$$

Plug into ODE to get:

$$\begin{aligned} x \left(-\frac{x \sin(x) + \cos(x)}{x^2}\right) + \frac{\cos(x)}{x} &= -\sin(x) - \frac{\cos(x)}{x} + \frac{\cos(x)}{x} \\ &= -\sin(x) \end{aligned}$$

Initial Condition:

$$y\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = 0$$

3. (40 points) Recall

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) (15 points) Using these definitions, show that

$$\cosh^2(x) + \sinh^2(x) = \cosh(2x)$$

Solutions:

$$\begin{aligned}\cosh^2(x) &= \frac{e^{2x} + e^{-2x} + 2}{4} \\ \sinh^2(x) &= \frac{e^{2x} + e^{-2x} - 2}{4} \\ \cosh^2(x) + \sinh^2(x) &= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4} \\ &= \cosh(2x)\end{aligned}$$

(b) (15 points) Find the length of the curve  $y = \cosh(x)$  from  $-\ln 2 \leq x \leq \ln 2$ .

Solutions:  $f(x) = \cosh(x)$  so  $f'(x) = \sinh(x)$

$$\begin{aligned}L &= \int_{-\ln 2}^{\ln 2} \sqrt{1 + (f'(x))^2} dx = \int_{-\ln 2}^{\ln 2} \sqrt{1 + \sinh^2(x)} dx \\ &= \int_{-\ln 2}^{\ln 2} \cosh(x) dx = 2 \int_0^{\ln 2} \cosh(x) dx \\ &= 2 \sinh(x) \Big|_0^{\ln 2} \\ &= \frac{3}{2} \quad (\sinh(\ln(2)) = 3/4)\end{aligned}$$

(c) (10 points) Find the surface area of the curve  $y = \cosh(x)$  from  $-\ln 2 \leq x \leq \ln 2$  rotated about the line  $y = 1$ . Leave your answer in terms of  $\ln 2$ .

Solutions:  $f(x) = \cosh(x)$  and  $f'(x) = \sinh(x)$

$$\begin{aligned}SA &= \int_{-\ln 2}^{\ln 2} 2\pi (f(x) - 1) \sqrt{1 + (f'(x))^2} dx = \int_{-\ln 2}^{\ln 2} 2\pi (\cosh(x) - 1) \sqrt{1 + \sinh^2(x)} dx \\ &= \int_{-\ln 2}^{\ln 2} 2\pi (\cosh^2(x) - \cosh(x)) dx = \int_{-\ln 2}^{\ln 2} 2\pi \left( \frac{\cosh(2x)}{2} + \frac{1}{2} - \cosh(x) \right) dx \\ &= \int_0^{\ln 2} 4\pi \left( \frac{\cosh(2x)}{2} + \frac{1}{2} - \cosh(x) \right) dx = 4\pi \left( \frac{\sinh(2x)}{4} + \frac{x}{2} - \sinh(x) \right) \Big|_0^{\ln 2} \\ &= 4\pi \left( \frac{15}{32} + \frac{\ln 2}{2} - \frac{3}{4} \right) \quad (\sinh(2 \ln(2)) = 15/8) \\ &= 2\pi \ln 2 - \frac{9\pi}{8}\end{aligned}$$

### Helpful Formulae

$$\sinh(2x) = 2 \sinh(x) \cosh(x)$$

$$\cosh(2x) = \sinh^2(x) + \cosh^2(x)$$

$$\cosh^2(x) = \frac{\cosh(2x)+1}{2}$$

$$\sinh^2(x) = \frac{\cosh(2x)-1}{2}$$

$$1 = \cosh^2(x) - \sinh^2(x)$$

$$e^{b \ln a} = a^b$$