APPM 1360 **Exam 2** **June 30, 2006**

1. (20 points) Find the volume of revolution for the region bounded by $\frac{2}{(x+1)(2-x)}$, the coordinate axes, and the line $x = 1$ rotate about the y-axis (**hint**: sketch the region first, then use shells). You can buy the shell integral for 5 points.
   Solutions: Using shells, the integral for the volume is:

$$2\pi \int_0^1 \frac{x}{(x+1)(2-x)} \mathrm{d}x$$

   Using partial fractions:

$$\begin{aligned} \frac{x}{(x+1)(2-x)} &= \frac{A}{x+1} + \frac{B}{2-x} \\ x &= A(2-x) + B(x+1) \\ x = -1 : A &= -\frac{1}{3} \\ x = 2 : B &= \frac{2}{3} \end{aligned}$$

   Then,

$$\begin{aligned} 2\pi \int_0^1 \frac{x}{(x+1)(2-x)} \mathrm{d}x &= 2\pi \int_0^1 \left(\frac{-1/3}{x+1} + \frac{2/3}{2-x} \right) \mathrm{d}x \\ &= 2\pi \left(-\frac{1}{3} \ln(x+1) - \frac{2}{3} \ln(2-x) \right)_0^1 \\ &= 2\pi \left(-\frac{1}{3} \ln(2) + \frac{2}{3} \ln(2) \right)_0^1 \\ &= \frac{2\pi \ln 2}{3} \end{aligned}$$

2. (30 points, 10 each) Calculate the following integrals.
   (a) $\int x^a \ln(x) \mathrm{d}x \qquad (a \neq -1)$
   Solutions: Let $u = \ln(x) \longrightarrow \mathrm{d}u = \frac{dx}{x}$. Let $\mathrm{d}v = x^a \mathrm{d}x \longrightarrow v = \frac{x^{a+1}}{a+1}$. Then, I.B.P. yields

$$\begin{aligned} \int x^a \ln(x) \mathrm{d}x &= \ln(x) \frac{x^{a+1}}{a+1} - \int \frac{x^{a+1}}{a+1} \frac{\mathrm{d}x}{x} \\ &= \ln(x) \frac{x^{a+1}}{a+1} - \frac{x^{a+1}}{(a+1)^2} + C \end{aligned}$$

   (b) $\int_1^e \frac{\mathrm{d}y}{y\sqrt{1+(\ln y)^2}}$
   Solutions: Let $u = \ln(y) \longrightarrow \mathrm{d}u = \frac{dy}{y}$. Then,

$$\int_1^e \frac{\mathrm{d}y}{y\sqrt{1+(\ln y)^2}} = \int_0^1 \frac{\mathrm{d}u}{\sqrt{1+u^2}}$$

   Let $u = \tan(\theta) \longrightarrow \mathrm{d}u = \sec^2(\theta)\mathrm{d}\theta$ and $\sqrt{1+u^2} = \sec(\theta)$. For the limits, $u = 0 \Longrightarrow \theta =$

0 and $u = 1 \Longrightarrow \theta = \frac{\pi}{4}$,

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}u}{\sqrt{1+u^2}} &= \int_0^{\frac{\pi}{4}} \sec(\theta)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \sec(\theta)\left(\frac{\sec(\theta)+\tan(\theta)}{\sec(\theta)+\tan(\theta)}\right)\mathrm{d}\theta \\
&= \ln|\sec(\theta)+\tan(\theta)|\Big|_0^{\frac{\pi}{4}} \\
&= \ln\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right)
\end{aligned}$$

(c) $\int_{\frac{2}{\sqrt{3}}}^{2} \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}}$
Solutions: Let $x = \sec(\theta) \longrightarrow \mathrm{d}x = \sec(\theta)\tan(\theta)\mathrm{d}\theta, \sqrt{x^2-1} = \tan(\theta)$. In the limits, $x = 2/\sqrt{3} \Longrightarrow \theta = \pi/6$ and $x = 2 \Longrightarrow \theta = \pi/3$. Then,

$$\begin{aligned}
\int_{\frac{2}{\sqrt{3}}}^{2} \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\mathrm{d}\theta}{\sec(\theta)} \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(\theta)\mathrm{d}\theta \\
&= \sin(\theta)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \frac{\sqrt{3}-1}{2}
\end{aligned}$$

3. (20 points, 10 each) Determine whether the following integrals converge or diverge by any method (integration, direct comparison test, limit comparison test).

(a) $\int_1^\infty \frac{\mathrm{d}x}{x^3+1}$
Solutions: Consider the function $\frac{1}{x^3}$. Since $\frac{1}{x^3+1} \le \frac{1}{x^3}$ and $\int_1^\infty \frac{\mathrm{d}x}{x^3}$ converges, the integral in question converges. Or,

$$\begin{aligned}
\lim_{x\longrightarrow\infty} \frac{\frac{1}{x^3}}{\frac{1}{x^3+1}} &= \lim_{x\longrightarrow\infty}\left(1+\frac{1}{x^3}\right) \\
&= 1
\end{aligned}$$

By, the limit comparison test, since the $\int_1^\infty \frac{\mathrm{d}x}{x^3}$ converges, then the integral in question does also.

(b) $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{e^x-x}}$
Solutions: Consider the function $\frac{1}{\sqrt{e^x}}$. By the limit comparison test,

$$\begin{aligned}
\lim_{x\longrightarrow\infty} \frac{\frac{1}{\sqrt{e^x}}}{\frac{1}{\sqrt{e^x-x}}} &= \lim_{x\longrightarrow\infty}\sqrt{\frac{e^x-x}{e^x}} \\
&= \lim_{x\longrightarrow\infty}\sqrt{1-\frac{x}{e^x}} \\
&= 1
\end{aligned}$$

Since $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{e^x}} = \frac{\sqrt{e}}{2}$, then the integral in question converges by the limit comparison test.

4. (30 points) Determine whether the following sequences converge or diverge as $n \to \infty$.

   (a) $a_n = \sinh(\ln n)$
   Solutions:

$$\begin{aligned} \sinh(\ln n) &= \frac{e^{\ln n} - e^{-\ln n}}{2} \\ &= \frac{n - \frac{1}{n}}{2} \qquad \longrightarrow \infty \quad \text{as } n \longrightarrow \infty \end{aligned}$$

   divergence

   (b) $a_n = \frac{\ln n^2}{\ln 2n}$
   Solutions

$$\begin{aligned} \frac{\ln n^2}{\ln 2n} &= \frac{2 \ln n}{\ln 2 + \ln n} \\ &= \frac{2}{1 + \frac{\ln 2}{\ln n}} \\ &= 2 \quad \text{as } n \longrightarrow \infty \end{aligned}$$

   convergence

   (c) $a_n = \left(\frac{n}{n+1}\right)^n$
   Solutions: take natural log of each side.

$$\begin{aligned} \ln(a_n) &= n \ln\left(\frac{n}{n+1}\right) \\ &\longrightarrow \infty \times 0 \qquad \text{indeterminant form, put } n \text{ in denominator} \\ \ln(a_n) &= \frac{\ln\left(\frac{n}{n+1}\right)}{1/n} \\ &\longrightarrow \frac{0}{0} \qquad \text{indeterminant form, use l'hopitals rule} \\ \lim_{n\to\infty} \ln(a_n) &= \lim_{n\to\infty} \frac{1/(n(n+1))}{-1/n^2} \\ &= \lim_{n\to\infty} -\frac{n}{n+1} \\ &= -1 \end{aligned}$$

   Thus, $\ln(a_n) \to -1$ implies $a_n \to e^{-1}$, i.e. convergence.

Useful formulae
$\sinh(x) = \frac{e^x - e^{-x}}{2}$
$\ln(a^b) = b \ln a$
$\ln(ab) = \ln(a) + \ln(b)$