Exam #1, APPM 1360
Fall 2006

(1) (a) $\frac{d}{dx}\ln(\cos x) = \frac{1}{\cos x}\frac{d}{dx}\cos x = -\frac{\sin x}{\cos x} = -\tan x$ //

(b) $\frac{d}{dx}\left(x^x\right) = \frac{d}{dx}\left(e^{x\ln x}\right) = e^{x\ln x}\frac{d}{dx}(x\ln x)$

$= e^{x\ln x}\left[\ln x + x\frac{1}{x}\right] = x^x(\ln x + 1)$ //

(c) $\int \sinh(kx)\,dx = \frac{1}{k}\cosh(kx) + C$ //

(d) $\int 2e^{2x}\cosh(2x)\,dx = \int 2e^{2x}\frac{1}{2}\left(e^{2x}+e^{-2x}\right)dx$

$= \int \left(e^{4x}+1\right)dx = \frac{1}{4}e^{4x} + x + C$ //

(2) (a) $\frac{dy}{dx} = e^x (y-1)^2 \Rightarrow \frac{dy}{(y-1)^2} = e^x dx \Rightarrow$

$$\int \frac{dy}{(y-1)^2} = \int e^x dx \Rightarrow -\frac{1}{y-1} = e^x + c \Rightarrow$$

$$y - 1 = -\frac{1}{e^x + c} \Rightarrow y = 1 - \frac{1}{e^x + c}$$

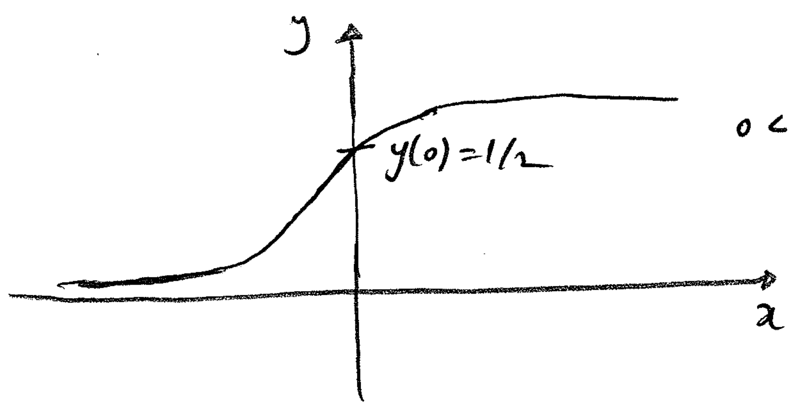
Initial condition: $y(0) = 1/2 \Rightarrow \frac{1}{2} = 1 - \frac{1}{c+1} \Rightarrow c = 1$

Finally $y = 1 - \frac{1}{e^x + 1} \Rightarrow y = \frac{e^x}{e^x + 1}$ //

(b) The function increases everywhere, since

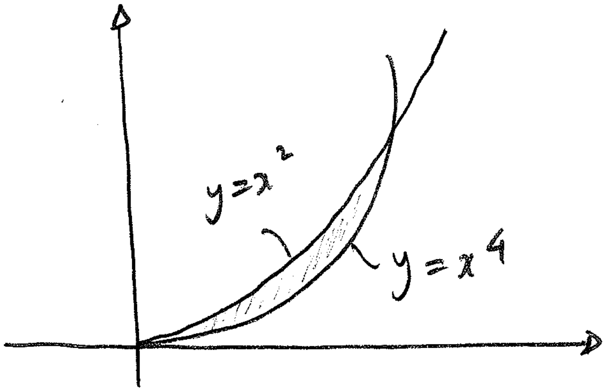
$\frac{dy}{dx} = e^x (y-1)^2$ is always positive and

$y \neq 1$.



$0 < y < 1$

(3)



Find where common points are : $x^2 = x^4 \Rightarrow$

$$\begin{cases} x=0 \\ x=1 \end{cases}$$

Since $0 \le x \le 1$ $\underline{x^2 > x^4}$

(a) Area $= \int_0^1 (x^2 - x^4)\, dx$ //

(b) Washers: Volume $= \int_0^1 \pi \left[R^2(y) - r^2(y) \right] dy$

$R(y) = 1 - \sqrt{y} = 1 - y^{1/2}$ // , $r(y) = 1 - \sqrt[4]{y} = 1 - y^{1/4}$ //

since we revolve around $x = 1$

(c) Shells: Volume $= \int_0^1 2\pi r h\, dx$

$r = 1 - x$ // , $h = x^2 - x^4$ //

(4) (a) Length $= \int_0^{\pi/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

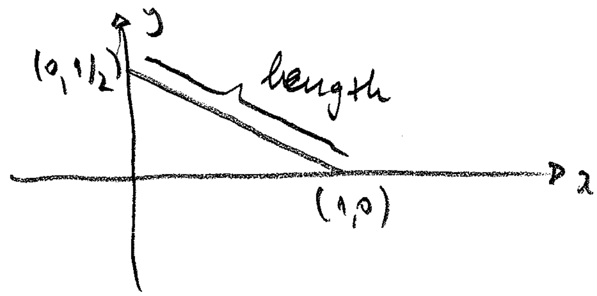
$$= \int_0^{\pi/4} \sqrt{4\sin^2(2t) + \underbrace{4\sin^2 t \cos^2 t}_{\sin^2(2t)}}\, dt$$

$\sin(2t) \geq 0$ when $0 \leq t \leq \pi/4$

$$= \int_0^{\pi/4} \sqrt{5}\, \sin(2t)\, dt = -\frac{\sqrt{5}}{2}\cos(2t)\Big|_0^{\pi/4} = \frac{\sqrt{5}}{2}$$ //

(b) $\left.\begin{array}{l} x = \cos(2t) \\ y = \sin^2 t \end{array}\right\}$ $\left.\begin{array}{l} x = 1 - 2\sin^2 t \\ y = \sin^2 t \end{array}\right\}$ $x = 1 - 2y \Rightarrow y = \frac{1}{2} - x/2$ //

straight line



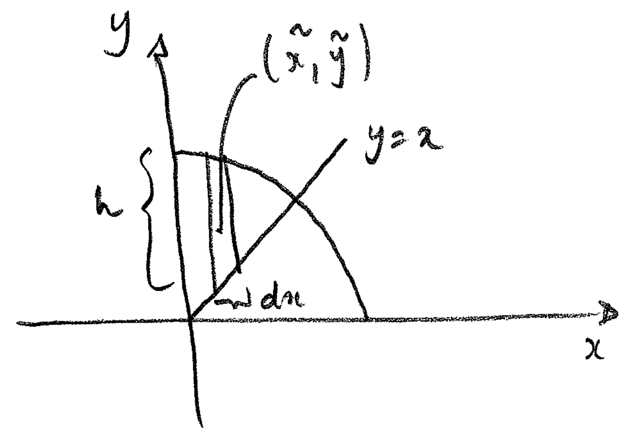
length $= \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{1}{4} + 1} = \sqrt{5}/2$ (Pythagora's theorem!)

(c) Surface $= \int_0^{\pi/4} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$

$$= 2\pi \int_0^{\pi/4} \sin^2 t\, \sqrt{5}\, \underbrace{\sin(2t)}_{2\sin t \cos t}\, dt = 4\pi\sqrt{5} \int_0^{\pi/4} \sin^3 t \cos t\, dt$$ //

(5) The centroid (constant density) is

$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \quad , \quad \bar{y} = \frac{\int \tilde{y}\, dm}{\int dm}$$



Find the common points

$$\left.\begin{matrix} x^2+y^2=a^2 \\ y=x \end{matrix}\right| \quad x^2+x^2=a^2 \Rightarrow \quad x = a\sqrt{2}/2 \qquad x>0$$

$$dm = \delta_0 \cdot \underbrace{h \cdot dx}_{dA} = \delta_0 \left(\sqrt{a^2-x^2} - x \right) dx$$

$$\tilde{x} = x \quad \text{and} \quad \tilde{y} = \underbrace{\frac{1}{2}\left[\text{circle} + \text{line} \right]}_{\text{geometrical average}} = \frac{1}{2}\left[\sqrt{a^2-x^2} + x \right]$$

Finally: $\text{Mass} = \int_0^{a\sqrt{2}/2} \delta_0 h\, dx = \delta_0 \int_0^{a\sqrt{2}/2} \left(\sqrt{a^2-x^2} - x \right) dx$ //

$$\int \tilde{x}\, dm = \int_0^{a\sqrt{2}/2} x\, \delta_0\, h\, dx = \delta_0 \int_0^{a\sqrt{2}/2} x \left(\sqrt{a^2-x^2} - x \right) dx \quad //$$

$$\int \tilde{y}\, dm = \int_0^{a\sqrt{2}/2} \tilde{y}\, \delta_0\, h\, dx = \delta_0 \int_0^{a\sqrt{2}/2} \frac{1}{2} \left[\underbrace{\left(\sqrt{a^2-x^2}\right)^2 - x^2}_{a^2-2x^2} \right] dx \quad //$$